THE COX RING OF AN ALGEBRAIC VARIETY WITH TORUS ACTION

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2000 *Mathematics Subject Classification.* 14C20, 14M25, 14J26.

ABSTRACT. We investigate the Cox ring of a normal complete variety X with algebraic torus action. Our first results relate the Cox ring of X to that of a maximal geometric quotient of X. As a consequence, we obtain a complete description of the Cox ring in terms of generators and relations for varieties with torus action of complexity one. Moreover, we provide a combinatorial approach to the Cox ring using the language of polyhedral divisors. Applied to smooth $\mathbb{K}^*$-surfaces, our results give a description of the Cox ring in terms of Orlik-Wagreich graphs. As examples, we explicitly compute the Cox rings of all Gorenstein del Pezzo $\mathbb{K}^*$-surfaces with Picard number at most two and the Cox rings of projectivizations of rank two vector bundles as well as cotangent bundles over toric varieties in terms of Klyachko's description.

1. INTRODUCTION

Let X be a normal complete algebraic variety defined over some algebraically closed field $\mathbb{K}$ of characteristic zero and suppose that the divisor class group

$\mathrm{Cl}(X)$ is finitely generated. The Cox ring of X is the graded $\mathbb{K}$-algebra

$$\mathcal{R}(X) \quad = \quad \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

see Section 2 for a detailed reminder. A basic problem is to present $\mathcal{R}(X)$ in terms of generators and relations. Besides the applications in number theory, see e.g. [9], the knowledge of generators and relations also opens a combinatorial approach to geometric properties of X, see [6] and [11].

In the present paper, we investigate the case that X comes with an effective algebraic torus action $T \times X \to X$. Our first result relates the Cox ring of X to that of a maximal orbit space of the T-action. For a point $x \in X$, denote by $T_x \subseteq T$ its isotropy group and consider the non-empty T-invariant open subset

$$X_0 \ := \ \{x \in X;\ \dim(T_x) = 0\} \ \subseteq \ X.$$

There is a geometric quotient $q\colon X_0 \to X_0/T$ with an irreducible normal but possibly non-separated orbit space X_0/T, see [22], and also for X_0/T one can define a Cox ring. Denote by $E_1, \ldots, E_m \subseteq X$ the (T-invariant) prime divisors supported in $X \setminus X_0$ and by $D_1, \ldots, D_n \subseteq X$ those T-invariant prime divisors who have a finite generic isotropy group of order $l_j > 1$. Moreover, let 1_{E_k} and 1_{D_j} denote the canonical sections of the divisors E_k and D_j respectively, and let $1_{q(D_j)} \in \mathcal{R}(X_0/T)$ be the canonical section of $q(D_j)$.

Theorem 1.1. *There is a graded injection $q^* \colon \mathcal{R}(X_0/T) \to \mathcal{R}(X)$ of Cox rings and the assignments $S_k \mapsto 1_{E_k}$ and $T_j \mapsto 1_{D_j}$ induce an isomorphism of $\mathrm{Cl}(X)$-graded rings*

$$\mathcal{R}(X) \quad \cong \quad \frac{\mathcal{R}(X_0/T)[S_1,\ldots,S_m,T_1,\ldots,T_n]}{\langle T_j^{l_j} - 1_{q(D_j)};\ 1 \le j \le n\rangle},$$

where $\mathrm{Cl}(X)$-grading on the right hand side is defined by associating to S_k the class of E_k and to T_j the class of D_j. In particular, $\mathcal{R}(X)$ is finitely generated if and only if $\mathcal{R}(X_0/T)$ is so.

If the dimension of T equals that of X, then our X is a toric variety, the subset $X_0 \subseteq X$ is the open T-orbit, the divisors $E_1,\ldots,E_m$ are the invariant prime divisors of X and there are no divisors D_j. Thus, for toric varieties, the above Theorem shows that the Cox ring is the polynomial ring in the canonical sections of the invariant prime divisors and hence gives the result obtained by D. Cox in [7].

The Cox ring $\mathcal{R}(X)$ can be further evaluated by using the fact that X_0/T admits a separation, i.e., a rational map $\pi\colon X_0/T \dashrightarrow Y$ to a variety Y, which is a local isomorphism in codimension one. After suitably shrinking, we may assume that there are prime divisors $C_0, \ldots, C_r$ on Y such that each inverse image $\pi^{-1}(C_i)$ is a disjoint union of prime divisors C_{ij}, where $1 \le j \le n_i$, the map π is an isomorphism over $Y \setminus (C_0 \cup \ldots \cup C_r)$ and all the D_j occur among the $D_{ij} := q^{-1}(C_{ij})$. Let $l_{ij} \in \mathbb{Z}_{\ge 1}$ denote the order of the generic isotropy group of D_{ij}.

Theorem 1.2. *There is a graded injection $\mathcal{R}(Y) \to \mathcal{R}(X)$ of Cox rings and the assignments $S_k \mapsto 1_{E_k}$ and $T_{ij} \mapsto 1_{q^{-1}(D_{ij})}$ induce an isomorphism of* $\mathrm{Cl}(X)$*-graded rings*

$$\mathcal{R}(X) \quad \cong \quad \frac{\mathcal{R}(Y)[S_1, \ldots, S_m, T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle T_i^{l_i} - 1_{C_i};\ 0 \le i \le r \rangle}.$$

where $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$, and the $\mathrm{Cl}(X)$*-grading on the right hand side is defined by associating to S_k the class of E_k and to T_{ij} the class of D_{ij}. In particular, $\mathcal{R}(X)$ is finitely generated if and only if $\mathcal{R}(Y)$ is so.*

Now suppose that the T-action on X is of complexity one, i.e., its biggest T-orbits are of codimension one in X. Then X_0/T is of dimension one and has a separation $\pi\colon X_0/T \to \mathbb{P}_1$. Choose $r \geq 1$ and $a_0, \ldots, a_r \in \mathbb{P}_1$ such that π is an isomorphism over $\mathbb{P}_1 \setminus \{a_0, \ldots, a_r\}$ and all the divisors D_j occur among the $D_{ij} := q^{-1}(y_{ij})$, where $\pi^{-1}(a_i) = \{y_{i1}, \ldots, y_{in_i}\}$. Let $l_{ij} \in \mathbb{Z}_{\geq 1}$ denote the order of the generic isotropy group of D_{ij}. For every $0 \leq i \leq r$, define a monomial

$$f_i \; := \; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \; \in \; \mathbb{K}[T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i].$$

Moreover, write $a_i = [b_i, c_i]$ with $b_i, c_i \in \mathbb{K}$ and for every $0 \leq i \leq r-2$ set $k = j+1 = i+2$ and define a trinomial

$$g_i \;\; := \;\; (c_k b_j - c_j b_k) f_i \; + \; (c_i b_k - c_k b_i) f_j \; + \; (c_j b_i - c_i b_j) f_k.$$

Theorem 1.3. *Let X be a normal complete variety with finitely generated divisor class group and an effective algebraic torus action $T \times X \to X$ of complexity one. Then, in terms of the data defined above, the Cox ring of X is given as*

$$\mathcal{R}(X) \;\; \cong \;\; \frac{\mathbb{K}[S_1, \ldots, S_m, T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i]}{\langle g_i;\ 0 \leq i \leq r-2 \rangle},$$

where 1_{E_k} *corresponds to* S_k*, and* $1_{D_{ij}}$ *to* T_{ij}*, and the* $\mathrm{Cl}(X)$*-grading on the right hand side is defined by associating to* S_k *the class of* E_k *and to* T_{ij} *the class of* D_{ij}*. In particular,* $\mathcal{R}(X)$ *is finitely generated.*

Note that finite generation of the Cox ring for a complexity one torus action with X_0/T rational may as well be deduced from [15].

In Section 4, we combine the results just presented with the description of torus actions in terms of polyhedral divisors given in [2] and [3] and that way obtain a combinatorial approach to the Cox ring, see Theorem 4.8. Similarly to the toric case [7], the advantage of the combinatorial treatment is that the divisor class group is easily accessible via the defining data and thus one has a simple approach to the grading of the Cox ring.

In Section 5, we give some applications. The description of the Cox ring given in Theorem 1.3 allows us to apply the language of bunched rings presented in [6] and [11] in order to investigate complete normal rational varieties X with a complexity one torus action. For example, in Corollary 5.2, we realize X as an invariant complete intersection in a toric variety

X', provided that any two points of X admit a common affine neighborhood. Moreover, in Corollary 5.3, we obtain explicit descriptions of the cone of movable divisor classes and the canonical divisor in terms of the divisors E_k and D_{ij}.

The first non-trivial examples of complexity one torus actions are complete normal rational $\mathbb{K}^*$-surfaces X. An important data is the Orlik-Wagreich graph associated to X, which describes the intersection theory of a canonical resolution $\widetilde{X}$ of X, see [17]. In Theorem 5.4, we show how to extract the Cox ring of $\widetilde{X}$ from the Orlik-Wagreich graph, which in turn allows to compute the Cox ring of X. In Theorem 5.6, we explicitly compute the Cox rings of all Gorenstein del Pezzo surfaces of Picard number at most two.

Finally, we consider in Section 5 projectivizations of equivariant vector bundles over complete toric varieties. We explicitly compute the Cox ring for the case of rank two bundles and for the case of the cotangent bundle over a smooth toric variety, see Theorems 5.7 and 5.9.

We would like to thank the referee for carefully reading the manuscript and for many helpful remarks and corrections.

2. Cox rings and universal torsors

Here we provide basic ingrediences for the proofs of Theorems 1.1 to 1.3, which also might be of independent interest. For example, in Proposition 2.3 we determine the Cox ring of a prevariety X in terms of that of a separation $X \to Y$ and Proposition 2.6 is a lifting statement for torus actions to the universal torsor in the case of torsion in the class group.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We will not only deal with varieties over $\mathbb{K}$ but more generally with prevarieties, i.e., possibly non-separated spaces. Recall that a ($\mathbb{K}$-)prevariety is a space X with a sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued functions such that $X = X_1 \cup \ldots \cup X_r$ holds with open subspaces X_i, each of which is an affine ($\mathbb{K}$-)variety.

In the sequel, X denotes an irreducible normal prevariety. As in the separated case, the group of Weil divisors is the free abelian group $\mathrm{WDiv}(X)$ generated by all prime divisors, i.e., irreducible subvarieties of codimension one. The divisor class group $\mathrm{Cl}(X)$ is the factor group of $\mathrm{WDiv}(X)$ modulo the subgroup of principal divisors. We define the Cox ring of X following [11, Sec. 2]. Suppose that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds and that the divisor class group $\mathrm{Cl}(X)$ is finitely generated. Let $\mathfrak{D} \subseteq \mathrm{WDiv}(X)$ be a finitely generated subgroup mapping onto $\mathrm{Cl}(X)$ and consider the sheaf of $\mathfrak{D}$-graded algebras

$$\mathcal{S} := \bigoplus_{D \in \mathfrak{D}} \mathcal{S}_D, \qquad \mathcal{S}_D := \mathcal{O}_X(D),$$

where multiplication is defined via multiplying homogeneous sections as rational functions on X. Let $\mathfrak{D}^0 \subseteq \mathfrak{D}$ be the kernel of $\mathfrak{D} \to \mathrm{Cl}(X)$. Fix a *shifting family*, i.e., a family of $\mathcal{O}_X$-module isomorphisms $\varrho_{D^0} \colon \mathcal{S} \to \mathcal{S}$, where $D^0 \in \mathfrak{D}^0$, such that

- $\varrho_{D^0}(\mathcal{S}_D) = \mathcal{S}_{D+D^0}$ for all $D \in \mathfrak{D}$, $D^0 \in \mathfrak{D}^0$,
- $\varrho_{D_1^0+D_2^0} = \varrho_{D_2^0} \circ \varrho_{D_1^0}$ for all $D_1^0, D_2^0 \in \mathfrak{D}^0$,
- $\varrho_{D^0}(fg) = f\varrho_{D^0}(g)$ for all $D^0 \in \mathfrak{D}^0$ and any two homogeneous f, g.

Consider the quasicoherent sheaf $\mathcal{I}$ of ideals of $\mathcal{S}$ generated by all sections of the form $f - \varrho_{D^0}(f)$, where f is homogeneous and D^0 runs through $\mathfrak{D}^0$. Then $\mathcal{I}$ is homogeneous with respect to the coarsened grading

$$\mathcal{S} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{S}_{[D]}, \qquad \mathcal{S}_{[D]} = \bigoplus_{D' \in D + \mathfrak{D}^0} \mathcal{O}_X(D').$$

Moreover, it turns out that $\mathcal{I}$ is a sheaf of radical ideals. Dividing the $\mathrm{Cl}(X)$-graded $\mathcal{S}$ by the homogeneous ideal $\mathcal{I}$, we obtain a quasicoherent sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras, the *Cox sheaf*: set $\mathcal{R} := \mathcal{S}/\mathcal{I}$, let $\pi\colon \mathcal{S} \to \mathcal{R}$ be the projection and define the grading by

$$\mathcal{R} = \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} = \pi\left(\mathcal{S}_{[D]}\right).$$

One can show that, up to isomorphism, the graded $\mathcal{O}_X$-algebra $\mathcal{R}$ does not depend on the choices of $\mathfrak{D}$ and the shifting family. We define the *Cox ring* $\mathcal{R}(X)$ of X, also called the *total coordinate ring* of X, to be the $\mathrm{Cl}(X)$-graded algebra of global sections of the Cox sheaf:

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}) \cong \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I}).$$

We are ready to perform first computations of Cox rings. Our aim is to relate the Cox ring of a prevariety X to that of a (separated) variety arising in a canonical way from X. We say that an open subset $U \subseteq X$ is *big* if the complement $X \setminus U$ is of codimension at least two in X.

Definition 2.1. By a *separation* of a prevariety X we mean a rational map $\varphi\colon X \dashrightarrow Y$ to a (separated) variety Y, which is defined on a big open subset $U \subseteq X$ and maps U locally isomorphic onto a big open subset $V \subseteq Y$.

Remark 2.2. Let $\varphi\colon X \dashrightarrow Y$ be a separation. Then there are big open subsets $U \subseteq X$ and $V \subseteq Y$ such that $\varphi\colon U \to V$ is a local isomorphism and moreover there are prime divisors $C_0, \ldots, C_r$ on V such that

(i) φ maps $U \setminus \varphi^{-1}(C_0 \cup \ldots \cup C_r)$ isomorphically onto $V \setminus (C_0 \cup \ldots \cup C_r)$,
(ii) Each $\varphi^{-1}(C_i)$ is a disjoint union of prime divisors C_{ij} of U.

As we will see in Proposition 3.5, every prevariety X with finitely generated divisor class group admits a separation $X \to Y$. Here comes how the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ are related to each other; for the sake of a simple notation, we identify prime divisors of the big open subsets $U \subseteq X$ and $V \subseteq Y$ with their closures in X and Y respectively.

Proposition 2.3. *Let $\varphi\colon X \dashrightarrow Y$ be a separation, $C_0, \ldots, C_r$ prime divisors on Y as in 2.2, and $\varphi^{-1}(C_i) =$*

$C_{i1} \cup \ldots \cup C_{in_i}$ with pairwise disjoint prime divisors C_{ij} on X. Then $\varphi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is injective, and we have

$$\mathrm{Cl}(X) \quad = \quad \varphi^* \,\mathrm{Cl}(Y) \,\oplus \bigoplus_{\substack{0 \le i \le r, \\ 1 \le j \le n_i - 1}} \mathbb{Z}[C_{ij}].$$

If $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds and $\mathrm{Cl}(X)$ is finitely generated, then there is a canonical injective pullback homomorphism $\varphi^*\colon \mathcal{R}(Y) \to \mathcal{R}(X)$ of Cox rings. Moreover, with $\deg(T_{ij}) := [C_{ij}]$ and $T_i := T_{i1} \cdots T_{in_i}$, the assignment $T_{ij} \mapsto 1_{C_{ij}}$ defines a $\mathrm{Cl}(X)$-graded isomorphism

$$\frac{\mathcal{R}(Y)[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle T_i - 1_{C_i};\ 0 \le i \le r \rangle} \quad \to \quad \mathcal{R}(X).$$

Proof. Since divisor class group and Cox ring do not change when passing to big open subsets, we may assume $U = X$ and $V = Y$ in the setting of Remark 2.2. The assertion on the divisor class group follows immediately from the facts that the principal divisors of X are precisely the pull backs of principal divisors on Y and that the divisor class group of X is generated by all pullback divisors and the classes $[C_{ij}]$, where $0 \le i \le r$ and $1 \le j \le n_i - 1$.

We turn to the Cox rings. Let $\mathfrak{D}_Y \subseteq \mathrm{WDiv}(Y)$ be a finitely generated subgroup containing $C_0, \ldots, C_r$ and mapping onto $\mathrm{Cl}(Y)$. Moreover, let $\mathfrak{D}_X \subseteq \mathrm{WDiv}(X)$ be the subgroup generated by $\varphi^*(\mathfrak{D}_Y)$ and the divisors C_{ij}, where $0 \le i \le r$ and $1 \le j \le n_i - 1$; note that $C_{in_i} \in \mathfrak{D}_X$ holds. Consider the associated graded sheaves

$$\mathcal{S}_Y := \bigoplus_{E \in \mathfrak{D}_Y} \mathcal{O}_Y(E), \qquad \mathcal{S}_X := \bigoplus_{D \in \mathfrak{D}_X} \mathcal{O}_X(D).$$

Then we have a graded injective pullback homomorphism $\varphi^* \colon \mathcal{S}_Y \to \mathcal{S}_X$, which in turn extends to a homomorphism

$$\psi \colon \mathcal{S}_Y[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i] \to \mathcal{S}_X, \qquad T_{ij} \mapsto 1_{C_{ij}}.$$

We show that ψ is surjective. Given a section h of $\mathcal{S}_X$ of degree $D \in \mathfrak{D}_X$, consider its divisor $D(h) = \mathrm{div}(h) + D$. If there occurs a $C_{ij} \in \mathfrak{D}_X$ in $D(h)$, then we may divide h in $\mathcal{S}_X$ by the corresponding $1_{C_{ij}}$. Doing this as often as possible, we arrive at some section h' of $\mathcal{S}_X$, homogeneous of some degree $D' \in \mathfrak{D}_X$, such that $D(h') = \mathrm{div}(h') + D'$ has no components C_{ij}. But then D' is a pullback divisor and h' is a pullback section. This in turn means that h' is a polynomial over $\varphi^*\mathcal{S}_Y$ and the $1_{C_{ij}}$.

Next, we determine the kernel of ψ, which amounts to determining the relations among the sections $s_{ij} := 1_{C_{ij}}$. Consider two coprime monomials F, F' in the

s_{ij} and two homogeneous pullback sections h, h' of $\varphi^*(\mathcal{S}_Y)$. If $\deg(hF) = \deg(h'F')$ holds in $\mathfrak{D}_X$, then the difference $\deg(F') - \deg(F)$ must be a linear combination of some $\varphi^*(C_i) \in \mathfrak{D}_X$ and hence F and F' are products of some $\varphi^* 1_{C_i}$. As a consequence, we obtain that any homogeneous (and hence any) relation among the s_{ij} is generated by the relations $T_i - 1_{C_i}$.

Finally, fix a shifting family ϱ_Y for $\mathcal{S}_Y$. Since $\mathfrak{D}_X^0 = \varphi^*(\mathfrak{D}_Y^0)$ holds, the pullback family $\varphi^*\varrho_Y$ extends uniquely to a shifting family ϱ_X for $\mathcal{S}_X$. We have $\mathcal{I}_X = \varphi^*(\mathcal{I}_Y)$ and hence obtain a well defined graded pullback homomorphism $\varphi^* \colon \mathcal{R}(Y) \to \mathcal{R}(X)$, which is injective, because $\varphi^* \colon \mathfrak{D}_Y/\mathfrak{D}_Y^0 \to \mathfrak{D}_X/\mathfrak{D}_X^0$ is so and $\varphi^* \colon \mathcal{S}_Y \to \mathcal{S}_X$ is an isomorphism when restricted to homogeneous components. Now one directly verifies that the above epimorphism ψ induces the desired isomorphism. $\square$

We apply this result to compute the Cox ring of the prevariety occurring as non-separated orbit space for torus actions of complexity one. Consider the projective line $\mathbb{P}_1$, a tuple $A = (a_0, \ldots, a_r)$ of pairwise different points a_i on $\mathbb{P}_1$, and a tuple $\mathfrak{n} = (n_0, \ldots, n_r) \in \mathbb{Z}_{\geq 1}^r$, where $r \geq 1$. Set

$$X_{ij} \ := \ \mathbb{P}_1 \setminus \bigcup_{k \neq i} a_k, \qquad 0 \leq i \leq r, \qquad 1 \leq j \leq n_i.$$

Then, gluing the X_{ij} along the common open subset $\mathbb{P}_1 \setminus \{a_0, \ldots, a_r\}$, one obtains an irreducible smooth prevariety $\mathbb{P}_1(A, \mathfrak{n})$ of dimension one. The inclusion maps $X_{ij} \to \mathbb{P}_1$ glue together to a morphism $\pi \colon \mathbb{P}_1(A, \mathfrak{n}) \to \mathbb{P}_1$, which is a separation. Writing a_{ij} for the point in $\mathbb{P}_1(A, \mathfrak{n})$ stemming from $a_i \in X_{ij}$, we obtain the fiber over a point $a \in \mathbb{P}_1$ as

$$\pi^{-1}(a) \quad = \quad \begin{cases} \{a_{i1}, \ldots, a_{in_i}\} & a = a_i \text{ for some } 0 \leq i \leq r, \\ \{a\} & a \neq a_i \text{ for all } 0 \leq i \leq r. \end{cases}$$

For every $0 \leq i \leq r$, define a monomial $T_i := T_{i1} \cdots T_{in_i}$ in the polynomial ring $\mathbb{K}[T_{ij};\ 0 \leq i \leq r,\ 1 \leq j \leq n_i]$. Moreover, for every $a_i \in \mathbb{P}_1$ fix a presentation $a_i = [b_i, c_i]$ with $b_i, c_i \in \mathbb{K}$ and for every $0 \leq i \leq r-2$ set $k = j + 1 = i + 2$ and define a trinomial

$$g_i \quad := \quad (c_k b_j - c_j b_k) T_i + (c_i b_k - c_k b_i) T_j + (c_j b_i - c_i b_j) T_k.$$

Proposition 2.4. *The divisor class group of* $\mathbb{P}_1(A, \mathfrak{n})$ *is free of rank* $n_0 + \ldots + n_r - r$ *and there is a decomposition*

$$\mathrm{Cl}(\mathbb{P}_1(A, \mathfrak{n})) \quad = \quad \bigoplus_{j=1}^{n_0} \mathbb{Z} \cdot [a_{0j}] \; \oplus \; \bigoplus_{i=1}^{r} \left(\bigoplus_{j=1}^{n_i - 1} \mathbb{Z} \cdot [a_{ij}] \right).$$

Moreover, in terms of the above data and with $\deg(T_{ij}) := [a_{ij}]$*, the Cox ring of* $\mathbb{P}_1(A, \mathfrak{n})$ *is given as*

$$\mathcal{R}(\mathbb{P}_1(A, \mathfrak{n})) \quad \cong \quad \frac{\mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle g_i;\ 0 \le i \le r-2 \rangle}.$$

Proof. The statement on the divisor class group is clear. The description of the Cox ring follows from Proposition 2.3 and the fact that the Cox ring $\mathcal{R}(\mathbb{P}_1)$ of the projective line is generated by the canonical sections $s_i := 1_{a_i}$ and has the relations

$$(c_k b_j - c_j b_k) s_i \; + \; (c_i b_k - c_k b_i) s_j \; + \; (c_j b_i - c_i b_j) s_k \quad = \quad 0,$$

where $0 \le i \le r-2$, and $k = j+1 = i+2$; note that the dependence of these relations on the choice of the b_i, c_i reflects the choice of a shifting family. □

Now we discuss some geometric aspects of the Cox ring. As before, let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group, and let $\mathcal{R}$ be a Cox sheaf. Suppose that $\mathcal{R}$ is locally of finite type; this holds for example if X is locally factorial or if $\mathcal{R}(X)$ is finitely generated. Then we may consider the relative spectrum

$$\widehat{X} \quad := \quad \mathrm{Spec}_X(\mathcal{R}).$$

The $\mathrm{Cl}(X)$-grading of the sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras defines an action of the diagonalizable group $H_X := \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X)]$ on $\widehat{X}$, and the canonical morphism $p\colon \widehat{X} \to X$ is a good quotient, i.e., it is an H_X-invariant affine morphism satisfying

$$\mathcal{O}_X \quad = \quad (p_*\mathcal{O}_{\widehat{X}})^{H_X}.$$

We call $p\colon \widehat{X} \to X$ the *universal torsor* associated to $\mathcal{R}$. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then we define the *total coordinate space* of X to be the affine variety $\overline{X} = \mathrm{Spec}(\mathcal{R}(X))$ together with the H_X-action defined by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$.

As usual, we say that a Weil divisor $\sum a_D D$, where D runs through the irreducible hypersurfaces, on a prevariety Y with an action of a group G is *G-invariant* if $a_D = a_{g\cdot D}$ holds for all $g \in G$. We say that Y is *G-factorial* if every G-invariant divisor on G is principal. Moreover, we say that a prevariety Y is of *affine intersection* if for any two affine open subsets $V, V' \subseteq Y$ the intersection $V \cap V'$ is again affine.

Proposition 2.5. *Let X be an irreducible smooth prevariety of affine intersection with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group. Let $\mathcal{R}$ be a Cox sheaf and denote by $p\colon \widehat{X} \to X$ the associated universal torsor.*

(i) *$\widehat{X}$ is a normal quasiaffine variety, and every homogeneous invertible function on $\widehat{X}$ is constant. If $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds or $\mathrm{Cl}(X)$ is free, then even every invertible function on $\widehat{X}$ is constant.*

(ii) *The action of H_X on $\widehat{X}$ is free and $\widehat{X}$ is H_X-factorial. If $\mathrm{Cl}(X)$ is free, then $\widehat{X}$ is even factorial.*

Proof. Normality of $\widehat{X}$ follows from [5, Lemma 3.10]. Since X is of affine intersection, it can be covered by open affine subsets, the complements of which are of pure codimension one. Together with smoothness this implies that X is divisorial in the sense of [5, Sec. 4]. Thus, we infer from [5, Prop. 6.3] that $\widehat{X}$ a quasiaffine variety. The fact that every homogeneous invertible function on $\widehat{X}$ is constant is seen as in [11, Prop. 2.2 (i)]. Moreover, [5, Thm. 7.3] tells us that every invertible function on $\widehat{X}$ is constant if we have

$\Gamma(X, \mathcal{O}) = \mathbb{K}$. For Assertion (ii), we can proceed exactly as in the proof of [11, Prop. 2.2 (iv)]. □

Proposition 2.6. *Let X be an irreducible smooth prevariety of affine intersection with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group. Let $\mathcal{R}$ be a Cox sheaf on X and $p \colon \widehat{X} \to X$ the associated universal torsor. Assume that $T \times X \to X$ is an effective algebraic torus action.*

(i) *There is a T-action on $\widehat{X}$ and an epimorphism $\varepsilon \colon T \to T$ such that for all $h \in H_X$, $t \in T$ and $z \in \widehat{X}$ one has*

$$t \cdot h \cdot z = h \cdot t \cdot z, \qquad p(t \cdot z) = \varepsilon(t) \cdot p(z).$$

If the divisor class group $\mathrm{Cl}(X)$ is free, then one may take the homomorphism $\varepsilon \colon T \to T$ to be the identity.

(ii) *Let $T \times H_X$ act on $\widehat{X}$ as in (i), let $G' \subseteq T \times H_X$ be the trivially acting subgroup and consider the induced effective action of $G := (T \times H_X)/G'$ on $\widehat{X}$. Then for any $z \in \widehat{X}$, there is an isomorphism of isotropy groups $G_z \cong T_{p(z)}$.*

Proof. We prove (i). Take a group $\mathfrak{D} \subseteq \mathrm{WDiv}(X)$ of Weil divisors mapping onto the divisor class group, and let $D_1, \ldots, D_r \in \mathrm{WDiv}(X)$ be a basis of $\mathfrak{D}$ such that the kernel $\mathfrak{D}_0 \subseteq \mathfrak{D}$ of $\mathfrak{D} \to \mathrm{Cl}(X)$ has a basis of the form $a_i D_i$, where $1 \le i \le s$ with some $s \le r$.

For every D_i choose a T-linearization, and via tensoring these linearizations, define a T-linearization of the whole group $\mathfrak{D}$, compare [10, Sec. 1]. Note that the T-linearization of the trivial divisor $a_i D_i$ is given by a character χ_i. Set $b := a_1 \cdots a_s$ and consider the epimorphism $\varepsilon \colon T \to T$, $t \mapsto t^b$. Then we have a new T-action

$$T \times X \ \to \ X, \qquad (t, x) \ \mapsto \ \varepsilon(t) \cdot x.$$

The divisors $D \in \mathfrak{D}$ are as well linearized with respect to this new action. Twisting each T-linearization of D_i with χ_i^{-b/a_i}, we achieve that each $a_i D_i$ is trivially T-linearized with respect to the new T-action on X. Thus, we may choose T-equivariant isomorphisms $\varrho_i \colon \mathcal{O}_X \to \mathcal{O}_X(a_i D_i)$.

Let $\mathcal{S}$ denote the $\mathfrak{D}$-graded sheaf defined by $\mathfrak{D}$. Using the isomorphisms ϱ_i, we construct a T-equivariant shifting family: for $D^0 = b_1 a_1 D_1 + \cdots + b_s a_s D_s$, define a T-equivariant isomorphism $\varrho_{D^0} \colon \mathcal{S} \to \mathcal{S}$ by sending a $\mathfrak{D}$-homogeneous f to

$$\varrho_{D^0}(f) \quad := \quad \varrho_1(1)^{b_1} \cdots \varrho_s(1)^{b_s} f.$$

The ideal $\mathcal{I}$ of $\mathcal{S}$ associated to this shifting family is T-homogeneous. This means that the T-action on $\mathrm{Spec}_X \mathcal{S}$ defined by the T-linearization of $\mathfrak{D}$ leaves $\widehat{X}$ invariant. By construction, the torsor $p\colon \widehat{X} \to X$ is T-equivariant, when we take the new T-action on X.

We turn to (ii). Let $\varepsilon\colon T \to T$ be as in (i). A first step is to show that for any given point $z \in \widehat{X}$, the kernel of ineffectivity $G' \subseteq T \times H_X$ can be written as

$$G' \quad = \quad \{(t,h) \in (T \times H_X)_z;\ \varepsilon(t) = 1\}.$$

In order to verify the inclusion "$\subseteq$", let $(t,h) \in G'$ be given. Then $(t,h)\cdot z' = z'$ holds for every point $z' \in \widehat{X}$. In particular, (t,h) belongs to $(T \times H_X)_z$. Moreover, we obtain $\varepsilon(t)\cdot p(z') = p(z')$ for every $z' \in \widehat{X}$. Since $p\colon \widehat{X} \to X$ is surjective and T acts effectively on X, this implies $\varepsilon(t) = 1$.

For checking the inclusion "$\supseteq$", consider $(t,h) \in (T \times H_X)_z$ with $\varepsilon(t) = 1$. Then, for every $z' \in \widehat{X}$, we have $p((t,h)\cdot z') = p(z')$. Consequently $t\cdot z' = h(t,z')\cdot z'$ holds with a uniquely determined $h(t,z') \in H_X$. Consider the assignment

$$\eta\colon \widehat{X} \to H_X, \qquad z' \mapsto h(t,z').$$

Since H_X acts freely we may choose for any z' homogeneous functions $f_1,\ldots,f_r$, defined near z' with

$f_i(z') = 1$ such that their weights $\chi_1, \ldots, \chi_r$ form a basis of the character group of H_X. Then, near z', we have a commutative diagram

$$\begin{array}{ccc} & \widehat{X} & \\ {}^{\eta}\swarrow & & \searrow^{z' \mapsto (f_1(t\cdot z'),\ldots,f_r(t\cdot z'))} \\ H_X & \xrightarrow[h' \mapsto (\chi_1(h'),\ldots,\chi_r(h'))]{\cong} & (\mathbb{K}^*)^r \end{array}$$

Consequently, the map η is is a morphism. Moreover, pulling back characters of H_X via η gives invertible H_X-homogeneous functions on $\widehat{X}$, which by Proposition 2.5 (i) are constant. Thus, η is constant, which means that $h(t) := h(t, z')$ does not depend on z'. By construction, $(t, h(t)^{-1})$ belongs to G'. Moreover, $t \cdot z = h^{-1} \cdot z$ and freeness of the H_X-action give $h(t) = h^{-1}$. This implies $(t, h) \in G'$.

We are ready to prove the assertion. Note that $(t, h) \mapsto \varepsilon(t)$ defines a homomorphism $\beta\colon (T \times H_X)_z \to T_{p(z)}$. We claim that β is surjective. Given $t \in T_{p(z)}$, choose $t' \in T$ with $\varepsilon(t') = t$. Then we have

$$p(t' \cdot z) \;=\; \varepsilon(t') \cdot p(z) \;=\; t \cdot p(z) \;=\; p(z).$$

Consequently, $t' \cdot z = h \cdot z$ holds for some $h \in H_X$. Thus, $(t', h^{-1}) \in (T \times H_X)_z$ is mapped by β to $t \in T_{p(z)}$. By the first step, the kernel of β is just G'. This gives a commutative diagram

$$\begin{array}{ccc} (T \times H_X)_z & \xrightarrow{\ \beta\ } & T_{p(z)} \\ & {\scriptstyle /G'} \searrow \quad \nearrow {\scriptstyle \cong} & \\ & G_z & \end{array}$$

□

In the sequel, we mean by a *universal torsor* for X more generally any good quotient $q\colon \mathcal{X} \to X$ for an action of H_X on a variety $\mathcal{X}$ such that there is an equivariant isomorphism $\imath\colon \mathcal{X} \to \widehat{X}$ with $q = p \circ \imath$.

Proposition 2.7. *Let $\mathcal{X}$ be a normal quasiaffine variety with a free action of a diagonalizable group H. If every invertible function on $\mathcal{X}$ is constant and $\mathcal{X}$ is H-factorial, then the quotient $q\colon \mathcal{X} \to X$ is a universal torsor for $X := \mathcal{X}/H$.*

Proof. We have $H = \operatorname{Spec} \mathbb{K}[K]$ with the character lattice K of H. A first step is to provide an isomorphism $K \to \operatorname{Cl}(X)$. Cover $\mathcal{X}$ by H-invariant affine open subsets W_j such that, for every $w \in K$ and every j, there is a w-homogeneous $h_{w,j} \in \Gamma(W_j, \mathcal{O}^*)$. Moreover, for every $w \in K$, fix a w-homogeneous $h_w \in \mathbb{K}(\mathcal{X})^*$. Then the H-invariant local equations $h_w/h_{w,j}$ define a Weil divisor $D(h_w)$ on X satisfying

$$D(h_w) = \mathrm{div}(h_w/h_{w,j}) \text{ on } q(W_j), \quad q^*(D(h_w)) = \mathrm{div}(h_w).$$

We claim that the assignment $w \mapsto D(h_w)$ induces an isomorphism from K onto $\mathrm{Cl}(X)$, not depending on the choice of h_w:

$$K \to \mathrm{Cl}(X), \qquad w \mapsto \overline{D}(w) := [D(h_w)].$$

To see that the class $\overline{D}(w)$ does not depend on the choice of h_w, consider a further w-homogeneous $g_w \in \mathbb{K}(\mathcal{X})^*$. Then $f := h_w/g_w$ is an invariant rational function descending to X, where we obtain

$$D(h_w) - D(g_w) = \mathrm{div}(f).$$

Thus, $K \to \mathrm{Cl}(X)$ is a well defined homomorphism. To verify injectivity, let $D(h_w) = \mathrm{div}(f)$ for some $f \in \mathbb{K}(X)^*$. Then we obtain $\mathrm{div}(h_w) = \mathrm{div}(q^*(f))$. Thus, $h_w/q^*(f)$ is an invertible homogeneous function on $\mathcal{X}$ and hence is constant. This implies $w = 0$. For surjectivity, let any $D \in \mathrm{WDiv}(X)$ be given. Then $q^*(D)$ is an H-invariant divisor on $\mathcal{X}$ and hence we have $q^*(D) = \mathrm{div}(h)$ with some rational function h on $\mathcal{X}$, which is homogeneous of some degree w. This means $D = D(h)$.

Now, choose a group $\mathfrak{D} \subseteq \mathrm{WDiv}(X)$ of Weil divisors mapping onto the divisor class group $\mathrm{Cl}(X)$, and let $D_1, \ldots, D_r \in \mathrm{WDiv}(X)$ be a basis of $\mathfrak{D}$ such that the kernel $\mathfrak{D}_0 \subseteq \mathfrak{D}$ of $\mathfrak{D} \to \mathrm{Cl}(X)$ is generated by multiples $a_i D_i$, where $1 \le i \le s$ with some $s \le r$. Set

$$\mathcal{S} := \bigoplus_{D \in \mathfrak{D}} \mathcal{S}_D, \qquad \mathcal{S}_D := \mathcal{O}_X(D).$$

By the preceding consideration, we may assume $D_i = D(h_{w_i})$ for $1 \le i \le r$. Then, for $D = b_1 D_1 + \ldots + b_r D_r$, we have $D = D(h_w)$ with $h_w = h_{w_1}^{b_1} \cdots h_{w_r}^{b_r}$, where $w = [D]$ is the class of $D = D(h_w)$ in $K = \mathrm{Cl}(X)$. For any open $U \subseteq X$, we have an isomorphism of $\mathbb{K}$-vector spaces

$$\Phi_{U,D} \colon \Gamma(U, \mathcal{O}(D(h_w))) \to \Gamma(q^{-1}(U), \mathcal{O})_w, \quad g \mapsto q^*(g) h_w.$$

In fact, on each $U_j := q(W_j) \cap U$ the section g is given as $g = g_j' h_j / h_w$ with a regular function $g_j' \in \mathcal{O}(U_j)$. Consequently, the function $q^*(g) h_w$ is regular on $q^{-1}(U)$. In particular, the assignment is a well defined homomorphism. Moreover, $f \mapsto f/h_w$ defines an inverse homomorphism.

The $\Phi_{U,D}$ fit together to an epimorphism of sheaves $\Phi \colon \mathcal{S} \to q_*(\mathcal{O}_{\mathcal{X}})$. We claim that the kernel $\mathcal{I}$ of Φ is the ideal of a shifting family ϱ. Indeed, for any $D^0 \in \mathfrak{D}^0$, consider

$$h^0 \ := \ \Phi_{X,D^0}^{-1}(\Phi_{X,0}(1)) \ \in \ \Gamma(X, \mathcal{S}_{D^0}).$$

Then $\varrho_{D_0} : \mathcal{S}_D \to \mathcal{S}_{D+D^0}$, $g \mapsto h^0 g$ is as wanted. Thus, we obtain an induced isomorphism $\mathcal{R} \to q_*\mathcal{O}_{\mathcal{X}}$, where $\mathcal{R} = \mathcal{S}/\mathcal{I}$ is the associated Cox sheaf. This in turn defines the desired isomorphism $\mathcal{X} \to \widehat{X}$. □

3. Proof of Theorems 1.1, 1.2 and 1.3

We begin with a couple of elementary observations. Let a diagonalizable group G act effectively on a normal quasiaffine variety U. Recall that a function $f \in \Gamma(U, \mathcal{O})$ is said to be G-homogeneous of weight $\chi \in \mathbb{X}(G)$ if one has $f(g{\cdot}x) = \chi(g)f(x)$ for all $g \in G$ and $x \in X$.

Lemma 3.1. *If there is a G-fixed point $x \in U$, then every G-homogeneous function $f \in \Gamma(U, \mathcal{O})$ with $f(x) \neq 0$ is G-invariant.*

Proof. Let $\chi \in \mathbb{X}(G)$ be the weight of $f \in \Gamma(U, \mathcal{O})$. Then, for every $g \in G$, we have $f(x) = \chi(g)f(x)$, which implies $\chi(g) = 1$. Thus, χ is the trivial character. □

By a G-prime divisor on U we mean a G-invariant Weil divisor $\sum a_D D$, where D runs through the prime divisors, we always have $a_D \in \{0, 1\}$ and G permutes transitively the D with $a_D = 1$. Let $B_1, \ldots, B_m \subseteq U$ be G-prime divisors and suppose that there are homogeneous functions $f_1, \ldots, f_m \in \Gamma(U, \mathcal{O})$ that satisfy $\mathrm{div}(f_i) = B_i$. Let $\chi_i \in \mathbb{X}(G)$ be the weight of f_i.

Lemma 3.2. *For $i = 1, \ldots, m$, let $G_i \subseteq G$ be the generic isotropy group of B_i and set $G_0 := G_1 \cdots G_m \subseteq G$.*

(i) *The restriction of χ_i to G_i generates the character group $\mathbb{X}(G_i)$.*
(ii) *For any two i, j with $j \neq i$, the function f_i is G_j-invariant.*
(iii) *The group G_0 is isomorphic to the direct product of the $G_i \subseteq G$.*
(iv) *$\Gamma(U, \mathcal{O})$ is generated by $f_1, \ldots, f_m$ and the G_0-invariant functions of U.*

Proof. Choose G-invariant affine open subsets $U_i \subseteq U$ such that $A_i := U_i \cap B_i$ is non-empty and $U_i \cap B_j$ is empty for every $j \neq i$.

To prove (i), let $\xi_i \in \mathbb{X}(G_i)$ be given. Then ξ_i is the restriction of some $\eta_i \in \mathbb{X}(G)$. Let $V_i \subseteq U_i$ be a G-invariant affine open subset on which G acts freely, and choose a non-trivial G-homogeneous function h_i

of weight η_i on V_i. Suitably shrinking U_i, we achieve that h_i is regular without zeroes on $U_i \setminus A_i$. Then, on U_i, the divisor $\mathrm{div}(h_i)$ is a multiple of the G-prime divisor $A_i = \mathrm{div}(f_i)$ and hence $h_i = a_i f_i^k$ holds with a G-homogeneous invertible function a_i on U_i and some $k \in \mathbb{Z}$. By Lemma 3.1, the function a_i is G_i-invariant. We conclude $\eta_i = k\chi_i$ on G_i.

Assertion (ii) is clear by Lemma 3.1. To obtain (iii), it suffices to show that χ_i is trivial on G_j for any two i, j with $j \neq i$. But, according to (ii), we have $f_i = \chi_i(g) f_i$ for every $g \in G_j$, which gives the claim. Finally, we verify (iv). Given a G-homogeneous function $f \in \Gamma(U, \mathcal{O})$, we may write $f = f' f_1^{\nu_1} \cdots f_m^{\nu_m}$ with $\nu_i \in \mathbb{Z}_{\geq 0}$ and a regular function f' on U, which is homogeneous with respect to some weight $\chi' \in \mathbb{X}(G)$ and has order zero along each G-prime divisor B_i. By Lemma 3.1, the function f' is invariant under every G_i and thus under G_0. □

Now we specialize to the case that $B_1, \ldots, B_m \subseteq U$ are precisely the G-prime divisors of U, which are contained in $U \setminus U_0$, where we set

$$U_0 \ := \ \{z \in U;\ \dim(G_z) = 0\} \ \subseteq \ U.$$

Proposition 3.3. *For $i = 1, \dots, m$, let $G_i \subseteq G$ be the generic isotropy group of B_i and set $G_0 := G_1 \cdots G_m \subseteq G$.*

(i) *Each G_i is a one-dimensional torus. Moreover, there is a non-empty G_0- invariant open subset $U' \subseteq U$ such that each B_i intersects the closure of any orbit $G_i \cdot z \subseteq U'$.*
(ii) *The G_0-action on U_0 is free, admits a geometric quotient $\pi_0 \colon U_0 \to V_0$ and the isotropy groups of the induced action of $H := G/G_0$ on V_0 satisfy $H_{\pi_0(x)} \cong G_x$ for every $x \in U_0$.*
(iii) *V_0 is quasiaffine and, moreover, if U is G-factorial (admits only constant invertible functions), then V_0 is H-factorial (admits only constant invertible functions).*
(iv) *Every G_0-invariant rational function of U has neither poles nor zeroes along outside U_0. Moreover, there is an isomorphism*

$$\Gamma(U_0, \mathcal{O})^{G_0}[S_1, \dots, S_m] \to \Gamma(U, \mathcal{O}), \qquad S_i \mapsto f_i.$$

Proof. We prove (i). By Lemma 3.2 (i), every G_i is a one-dimensional torus. To proceed, take any G_0-equivariant affine closure $U \subseteq \overline{U}$ and consider the quotient $\pi_i \colon \overline{U} \to \overline{U} /\!\!/ G_i$. It maps the fixed point set of the G_i-action isomorphically onto its image in the quotient space $\overline{U}/\!\!/G_i$. Since $\overline{U}/\!\!/G_i$ is irreducible and of dimension at most $\dim(\overline{U}) - 1$, we obtain $\pi_i(\overline{B}_i) = \overline{U}/\!\!/G_i$ for the closure $\overline{B}_i$ of B_i in $\overline{U}$. It follows that $\overline{B}_i$ is irreducible, equals the whole fixed point set of G_i in $\overline{U}$ and any G_i-orbit of $\overline{U}$ has a point of $\overline{B}_i$ in its closure.

We turn to (ii). Since none of the f_i has a zero inside U_0, we infer from Lemma 3.2 that G_0 acts freely on U_0. In particular, the action of G_0 on U_0 admits a geometric quotient $\pi_0 \colon U_0 \to V_0$ with a prevariety V_0. The statement on the isotropy groups of the H-action on V_0 is obvious.

We prove the statements made in (iii) and (iv). Denoting by $\mathbb{T}^m$ the standard m-torus $(\mathbb{K}^*)^m$, we have a well defined morphism of normal prevarieties

$$\varphi \colon U_0 \to V_0 \times \mathbb{T}^m, \qquad x \mapsto (\pi_0(x), f_1(x), \ldots, f_m(x)).$$

According to Lemma 3.2, the weight χ_i of f_i generates the character group of G_i for $i = 1, \ldots, m$. Using this and the fact that G_0 is the direct product of $G_1, \ldots, G_m$, we conclude that φ is bijective and thus an isomorphism. In particular, we conclude that V_0 is quasiaffine, because U and hence U_0 is so.

Now, endow V_0 with the induced action of $H = G/G_0$ and $\mathbb{T}^m$ with the diagonal G_0-action given by the weights $\chi_1, \ldots, \chi_m$ of $f_1, \ldots, f_m$. Then φ becomes G-equivariant, where G acts via the splitting $G = H \times G_0$ on $V_0 \times \mathbb{T}^m$. Using this, we see that G-factoriality of U_0 implies H-factoriality of V_0.

We show now that every G_0-invariant rational function $f \in \mathbb{K}(U_0)$ has neither zeroes nor poles outside U_0. Recall that $U \setminus U_0$ is the union of the zero sets B_i of f_i, which in turn are the fixed point sets of the G_i-actions on U. Since the general orbit $G_0 \cdot x \subseteq U$ has a point $x_i \in B_i$ in its closure, we see that f has neither poles nor zeroes along the B_i. In particular, if f is regular on U_0 then it is so on the whole U. As a consequence, we obtain that every invertible function on V_0 is constant provided the same holds for U.

Finally, according to Lemma 3.2 (iv), the homomorphism of (iv) is surjective. Moreover, since the weights $\chi_1, \ldots, \chi_m$ of the $f_1, \ldots, f_m$ are a basis of the character group of G_0, there are no relations among the f_i. □

Let a diagonalizable group H act effectively with at most finite isotropy groups on a quasiaffine variety V. Suppose that V is H-factorial and admits only constant invertible functions. Denote by $C_1, \ldots, C_n \subseteq V$ those H-prime divisors of V, on which H acts with a non trivial generic isotropy group H_j of order $l_j > 1$ and let $g_1, \ldots, g_n$ be homogeneous functions on V with $\mathrm{div}(g_j) = C_j$.

Proposition 3.4. *Consider the action of $H_0 := H_1 \cdots H_n \subseteq H$ on V, and let $\kappa \colon V \to W$ be the associated quotient.*

(i) *W is a quasiaffine variety with an effective induced action of H/H_0, and $h \mapsto \kappa^*(h)$ and $T_j \mapsto g_j$ define an isomorphism*

$$\Gamma(W, \mathcal{O})[T_1, \ldots, T_n]/\langle T_j^{l_j} - g_j^{l_j};\ j = 1, \ldots, n\rangle \ \to\ \Gamma(V, \mathcal{O}).$$

(ii) *W admits an (H/H_0)-factorial big open subset $W_0 \subseteq W$ such that H/H_0 acts freely on W_0 and W_0 has only constant invertible functions.*

Proof. We prove (i). By Lemma 3.2 (i), every H_j is cyclic. Moreover, Lemma 3.2 (iv) tells us that there is an epimorphism

$$\Gamma(W, \mathcal{O})[T_1, \ldots, T_n] \to \Gamma(V, \mathcal{O}), \quad h \mapsto \kappa^*(h), \quad T_j \mapsto g_j.$$

From Lemma 3.2 (iii) we infer that $H_0 \subseteq H$ is the direct product of $H_1, \ldots, H_n \subseteq H$. Thus, the quotient $\kappa \colon V \to W$ can as well be obtained by dividing stepwise by effective actions of the H_j. Using this, one directly checks that the kernel of this epimorphism is the ideal generated by $T_j^{l_j} - g_j^{l_j}$, where $1 \le j \le n$.

We turn to (ii). Note that W admits only constant invertible functions. Let $V_0 \subseteq V$ denote the subset consisting of all points $y \in V$ that have either trivial isotropy group $H_{0,y}$ or belong to some C_j and have isotropy group $H_{0,y} = H_j$. Note that $V_0 \subseteq V$ is big, H-invariant and open. Set $W_0 := \kappa(V_0)$. Then $W_0 \subseteq W$ is big and the restriction $\kappa\colon V_0 \to W_0$ is a quotient for the action of H_0. By construction, H/H_0 acts freely on W_0.

We show that W_0 is H/H_0-factorial. Since V_0 and W_0 are normal, there is a smooth (H/H_0)-invariant big open subset $W_1 \subseteq W_0$ such that $V_1 := \kappa^{-1}(W_1)$ is also smooth and big in V_0. We have to show that every (H/H_0)-linearizable bundle on W_1 is trivial. According to [14, Cor. 5.3], we have two exact sequences fitting into the following diagram

$$
\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Pic}(W_1) & & \\
 & & & & \downarrow {\scriptstyle \kappa^*} & & \\
1 & \longrightarrow & \mathbb{X}(H_0) & \xrightarrow{\alpha} & \mathrm{Pic}_{H_0}(V_1) & \xrightarrow{\beta} & \mathrm{Pic}(V_1) \\
 & & & & \downarrow {\scriptstyle \delta} & & \\
 & & & & \prod_{i=1}^{n} \mathbb{X}(H_i) & &
\end{array}
$$

where the isotropy groups $H_1, \ldots, H_n$ generate H_0 and hence $\beta \circ \kappa^*$ and $\delta \circ \alpha$ are injective. Given an (H/H_0)-linearized bundle L on W_1, the pullback $\kappa^*(L)$ is trivial by assumption, which means $\beta(\kappa^*(L)) = 1$. Consequently, L is trivial. □

Proof. Proof of Theorem 1.1 We prove the statement more generally for the cases that $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds or that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds and $\mathrm{Cl}(X)$ is free. Since the Cox ring of X and that of its smooth locus coincide, we may assume that X is smooth. Consider the universal torsor $p \colon \widehat{X} \to X$. By Proposition 2.5, this is a geometric quotient for a free action of the diagonalizable group $H_X := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ on $\widehat{X}$ and $\widehat{X}$ has only constant globally invertible functions. Fix a lifting of the T-action to $\widehat{X}$ as in Proposition 2.6 (i) and, as in Proposition 2.6 (ii), let G be the quotient of $T \times H_X$ by the kernel of ineffectivity of its action on $\widehat{X}$. Then G acts effectively on $\widehat{X}$ and Proposition 2.5 (ii) tells us that $\widehat{X}$ is G-factorial.

Consider the T-invariant prime divisors $E_1, \ldots, E_m \subseteq X$ supported in $X \setminus X_0$. Their inverse images $\widehat{E}_k := p^{-1}(E_k)$ are G-prime divisors and, since $\widehat{X}$ is G-factorial, we have $\widehat{E}_k = \operatorname{div}(f_k)$ with some G-homogeneous $f_i \in \mathcal{R}(X)$. According to Proposition 2.6 (ii), the $\widehat{E}_k$ are precisely the G-prime divisors supported in $\widehat{X} \setminus \widehat{X}_0$. Moreover, consider the T-invariant prime divisors $D_1, \ldots, D_n \subseteq X$ along which T acts with a finite generic isotropy group of order $l_j > 1$ and their (G-prime) inverse images $\widehat{D}_j := p^{-1}(D_j)$. As before, we see that $\widehat{D}_j = \operatorname{div}(g_j)$ holds with some G-homogeneous $g_j \in \mathcal{R}(X)$ and the generic isotropy group of the G-action on $\widehat{D}_j$ has order l_j. Note that none of the $\widehat{D}_j$ equals one of the $\widehat{E}_k$. Moreover, we may view the functions f_k and g_j as the canonical sections of the divisors E_k and D_j.

Let $G_k \subseteq G$ denote the generic isotropy group of $\widehat{E}_k \subseteq \widehat{X}_k$. The action of $G_0 := G_1 \cdots G_m$ on $\widehat{X}_0 = p^{-1}(X_0)$ admits a geometric quotient $\jmath_0 \colon \widehat{X}_0 \to \widehat{Y}_0$. The factor group $H := G/G_0$ acts with at most finite isotropy groups on $\widehat{Y}_0$ and, by Proposition 3.3 (ii), has generic isotropy group $H_j \subseteq H$ of order l_j along the divisors $\widehat{C}_j := \jmath_0(\widehat{D}_j)$. Set $H_0 := H_1 \cdots H_n$ and let $\kappa \colon \widehat{Y}_0 \to \widehat{Z}_0$ denote the quotient for the action of H_0 on $\widehat{Y}_0$. The induced action of $F := H/H_0$ on $\widehat{Z}_0$ admits again a geometric quotient $\widehat{Z}_0 \to Z_0$ and the whole situation fits into the following commutative diagram.

(3.1)
$$\begin{array}{ccccccc}
\widehat{X} & \supseteq & \widehat{X}_0 & \xrightarrow{/G_0} & \widehat{Y}_0 & \xrightarrow{/H_0} & \widehat{Z}_0 \\
\downarrow{\scriptstyle /H_X} & & \downarrow{\scriptstyle /H_X} & & & & \downarrow{\scriptstyle /F} \\
X & \supseteq & X_0 & & \xrightarrow[/T]{\qquad\qquad} & & Z_0
\end{array}$$

Replacing Z_0 and X_0 as well as $\widehat{Z}_0$, $\widehat{Y}_0$ and $\widehat{X}_0$ with suitable big open subsets, we achieve that the group F acts freely on $\widehat{Z}_0$.

We show that $\widehat{Z}_0 \to Z_0$ is a universal torsor for Z_0. According to Proposition 2.7 this means to verify that $\widehat{Z}_0$ is an F-factorial quasiaffine variety with only constant invertible functions. Proposition 3.3 provides the corresponding properties for the H-variety $\widehat{Y}_0$. Moreover, by Lemma 3.1, every g_j is G_0-invariant, hence g_j descends to a function on $\widehat{Y}_0$, where we have $\mathrm{div}(g_j) = \widehat{C}_j$. Thus, we can apply Proposition 3.4 to obtain the desired properties for $\widehat{Z}_0$ and the action of F.

The final task is to relate the Cox rings $\mathcal{R}(X) = \Gamma(\widehat{X}, \mathcal{O})$ and $\mathcal{R}(Z_0) = \Gamma(\widehat{Z}_0, \mathcal{O})$ to each other. Note that we have canonical inclusions of graded algebras

$$\Gamma(\widehat{X}, \mathcal{O}) \supseteq \Gamma(\widehat{X}_0, \mathcal{O})^{G_0} = \Gamma(\widehat{Y}_0, \mathcal{O}) \supseteq \Gamma(\widehat{Y}_0, \mathcal{O})^{H_0} = \Gamma(\widehat{Z}_0, \mathcal{O}),$$

where the first one is due to Proposition 3.3. This allows us in particular to view $\Gamma(\widehat{Z}_0, \mathcal{O})$ as a graded subalgebra of $\Gamma(\widehat{X}, \mathcal{O})$. The assertion now follows from Proposition 3.3 (iv) and Proposition 3.4 (i). □

In the above proof, we realized a big open subset of X_0/T as a quotient of a quasiaffine variety with only constant invertible functions by a free action of a diagonalizable group, see the diagram 3.1. According to Proposition 2.7, this allowed us to define a Cox ring for X_0/T. Moreover, we use this now to show that X_0/T admits a separation.

Proposition 3.5. *Let $\mathcal{X}$ be a normal quasiaffine variety with a free action of a diagonalizable group H. Suppose that every invertible function on $\mathcal{X}$ is constant and that $\mathcal{X}$ is H-factorial. Then $X := \mathcal{X}/H$ admits a separation.*

Proof. We first treat the case of a certain toric variety. Consider the standard action of $\mathbb{T}^r = (\mathbb{K}^*)^r$ on $\mathbb{K}^r$, let $\mathcal{Z} \subseteq \mathbb{K}^r$ be the union of all orbits of the big torus $\mathbb{T}^r \subseteq \mathbb{K}^r$ of dimension at least $r-1$, and let $H \subseteq \mathbb{T}^r$ be a closed subgroup acting freely on $\mathcal{Z}$. The fan Σ of $\mathcal{Z}$ has the extremal rays of the positive orthant $\mathbb{Q}^r_{\geq 0}$ as its maximal cones and $Z := \mathcal{Z}/H$ is the toric prevariety obtained by gluing the orbit spaces $\mathcal{Z}_\varrho/H$ along their common big torus T/H, where $\mathcal{Z}_\varrho \subseteq \mathcal{Z}$ denotes the affine toric chart corresponding to $\varrho \in \Sigma$. The embedding $H \to \mathbb{T}^r$ corresponds to a surjection $\mathbb{Z}^r \to K$ of the respective character groups. Let $P\colon \mathbb{Z}^r \to N$ be a map having $\mathrm{Hom}(K, \mathbb{Z})$ as its kernel. Then we obtain a canonical separation $Z \to Z'$ onto a toric variety Z', the fan of which lives in N and consists of the cones $P(\varrho)$, where $\varrho \in \Sigma$.

In the general case, choose a finitely generated graded subalgebra $A \subseteq \Gamma(\mathcal{X}, \mathcal{O})$ such that we obtain an open embedding $\mathcal{X} \subseteq \overline{X}$, where $\overline{X} := \operatorname{Spec} A$. Properly enlarging A, we may assume that it admits a system $f_1, \ldots, f_r$ of homogeneous generators such that each $\operatorname{div}(f_i)$ is H-prime in $\widehat{X}$. Consider the H-equivariant closed embedding $\overline{X} \to \mathbb{K}^r$ defined by $f_1, \ldots, f_r$ and let $\mathcal{Z} \subseteq \mathbb{K}^r$ be as above. By construction, $\mathcal{U} := \mathcal{Z} \cap \mathcal{X}$ is a big H-invariant open subset of $\mathcal{X}$, and we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{Z} \\ {\scriptstyle /H}\downarrow & & \downarrow{\scriptstyle /H} \\ U & \longrightarrow & Z \end{array}$$

where the induced map $U \to Z$ of quotients is a locally closed embedding and Z is a toric prevariety. Again by construction, the intersection of the invariant prime divisors of Z with U are prime divisors on U. Consequently, the restriction of $Z \to Z'$ defines the desired separation $U \to U'$ $\square$

Proof. Proof of Theorem 1.2 We prove the statement more generally for the cases that $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds or that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds and $\operatorname{Cl}(X)$ is free. By Proposition 3.5, the orbit space X_0/T admits a separation

$\pi\colon X_0/T \to Y$. According to Remark 2.2, we may assume that there are prime divisors $C_0, \ldots, C_r$ on Y such that each $\pi^{-1}(C_i)$ is a disjoint union of prime divisors $C_{ij} \subseteq X_0/T$, where $1 \le j \le n_i$, the map π is an isomorphism over $Y \setminus (C_0 \cup \ldots \cup C_r)$ and all the D_j occur among the divisors $D_{ij} := q^{-1}(C_{ij})$. Then, according to Proposition 2.3, we have

$$\mathcal{R}(X_0/T) \quad \cong \quad \frac{\mathcal{R}(Y)[\widetilde{T}_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle \widetilde{T}_i - 1_{C_i};\ 0 \le i \le r \rangle},$$

where the variables $\widetilde{T}_{ij}$ correspond to the canonical sections $1_{C_{ij}}$ and we define $\widetilde{T}_i := \widetilde{T}_{i1} \cdots \widetilde{T}_{in_i}$. Let $l_{ij} \in \mathbb{Z}_{\ge 1}$ denote the order of the generic isotropy group of $D_{ij} = q^{-1}(C_{ij})$. Then, by Theorem 1.1, we have

$$\mathcal{R}(X) \quad \cong \quad \mathcal{R}(X_0/T)[S_1, \ldots, S_m;\ T_{ij}] \;/\; \langle T_{ij}^{l_{ij}} - 1_{C_{ij}} \rangle,$$

where the variables T_{ij} correspond to the canonical sections $1_{C_{ij}}$; note that $1_{C_{ij}}$ and $1_{D_{ij}}$ are identified for $l_{ij} = 1$. Putting these two presentations of Cox ring together, we arrive at the assertion. □

Proof. Proof of Theorem 1.3 We prove the statement more generally for the case that $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds. Since the T-action on X is of complexity one, the orbit space X_0/T is of dimension one and smooth.

Moreover, using the diagram 3.1 and Proposition 3.3, we see that X_0/T admits only constant global functions and has a finitely generated divisor class group. It follows that X_0/T is isomorphic to $\mathbb{P}_1(A, \mathfrak{n})$, with $A = (a_0, \dots, a_r)$ and $\mathfrak{n} = (n_0, \dots, n_r)$ defined as in Theorem 1.3. By Proposition 2.4, the Cox ring $\mathbb{P}_1(A, \mathfrak{n})$ is given by

$$(3.2) \quad \mathcal{R}(\mathbb{P}_1(A, \mathfrak{n})) \quad \cong \quad \mathbb{K}[\widetilde{T}_{ij}] \ / \ \langle \widetilde{g}_i; \ 0 \le i \le r-2 \rangle,$$

where the variables $\widetilde{T}_{ij}$ correspond to the canonical sections of points a_{ij} in $X_0/T \cong \mathbb{P}_1(A, \mathfrak{n})$. Their inverse images $D_{ij} = q^{-1}(a_{ij})$ under $q \colon X_0 \to X_0/T$ are prime divisors with generic isotropy group of order $l_{ij} \ge 1$; note that $l_{ij} = 1$ is allowed. Applying Theorem 1.1 gives

$$(3.3) \quad \mathcal{R}(X) \quad \cong \quad \mathcal{R}(\mathbb{P}_1(A, \mathfrak{n}))[S_1, \dots, S_m, T_{ij}] \ / \ \langle T_{ij}^{l_{ij}} - \widetilde{T}_{ij} \rangle,$$

where the variables S_i correspond to the divisors of X having a one-dimensional generic isotropy group, the variables T_{ij} are the canonical sections of the divisors D_{ij}, and the $\widetilde{T}_{ij}$ are identified with their pullbacks under $X_0 \to X_0/T$; note that the pullback $q^*(a_{ij})$ equals $l_{ij} D_{ij}$. Now, putting the descriptions (3.2) and (3.3) together gives the assertion. □

Remark 3.6. Note that for factorial affine varieties with a complexity one torus action, D. Panyshev observed in [18, Remark 2.12] a presentation of the algebra of global functions by generators and trinomial relations.

4. Cox ring via polyhedral divisors

In this section, we combine Theorem 1.2 with the description of algebraic torus actions in terms of polyhedral divisors presented in [2] and [3] and provide a combinatorial approach to the Cox ring of an algebraic variety with torus action. We begin with a brief reminder on the language of polyhedral divisors.

In the sequel, N is a free finitely generated abelian group, and $M = \mathrm{Hom}(N, \mathbb{Z})$ is its dual. The associated rational vector spaces are denoted by $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$. Moreover, $\sigma \subseteq N_{\mathbb{Q}}$ is a pointed convex polyhedral cone, and $\omega \subseteq M_{\mathbb{Q}}$ is its dual cone. The relative interior of σ is denoted by σ°, and if τ is a face of σ, then we write $\tau \preceq \sigma$.

We consider convex polyhedra $\Delta \subseteq N_{\mathbb{Q}}$ admitting a decomposition $\Delta = \Pi + \sigma$ with a (bounded) polytope $\Pi \subseteq N_{\mathbb{Q}}$; we refer to σ as the *tail cone* of Δ and refer to

Δ as a *σ-polyhedron*. With respect to Minkowski addition, the set $\mathrm{Pol}^+_\sigma(N)$ of all σ-polyhedra is a monoid with neutral element σ. We consider also the empty set as an element of $\mathrm{Pol}^+_\sigma(N)$ and set $\Delta+\emptyset := \emptyset+\Delta := \emptyset$.

We are ready to enter the description of affine varieties with an action of the torus $T = \mathrm{Spec}\,\mathbb{K}[M]$. Let Y be a normal variety and fix a pointed convex polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. A *polyhedral divisor* on Y is a formal finite sum

$$\mathcal{D} \quad = \quad \sum_Z \Delta_Z \cdot Z,$$

where Z runs over the prime divisors of Y and the coefficients Δ_Z belong to $\mathrm{Pol}^+_\sigma(N)$; finiteness of the sum means that only finitely many coefficients Δ_Z differ from the tail cone σ.

The *locus* of a polyhedral divisor $\mathcal{D}$ on Y is the open subset $Y(\mathcal{D}) \subseteq Y$ obtained by removing all prime divisors $Z \subseteq Y$ with $\Delta_Z = \emptyset$. For every $u \in \omega \cap M$ we have the evaluation

$$\mathcal{D}(u) \quad := \quad \sum_Z \min_{v \in \Delta_Z} \langle u, v\rangle \cdot Z,$$

which is an ordinary rational divisor living on $Y(\mathcal{D})$. We call the polyhedral divisor $\mathcal{D}$ on Y *proper* if its locus is semiprojective, i.e., projective over some affine variety, and its evaluations $\mathcal{D}(u)$, where $u \in \omega \cap M$, have the following properties

(i) $\mathcal{D}(u)$ has a base point free multiple,
(ii) $\mathcal{D}(u)$ is big for $u \in \omega^{\circ} \cap M$.

Remark 4.1. Suppose that we have $Y = \mathbb{P}_n$, and consider a polyhedral divisor $\mathcal{D} = \sum \Delta_Z{\cdot}Z$. The *degree* of $\mathcal{D}$ is the polyhedron

$$\deg(\mathcal{D}) \ := \ \sum_Z \Delta_Z \cdot \deg(Z) \ \in \ \mathrm{Pol}^+_\sigma(N).$$

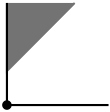
It provides a simple criterion for properness: if $\deg(\mathcal{D})$ is a proper subset of the tail cone of $\mathcal{D}$, then $\mathcal{D}$ is a proper polyhedral divisor, see [2, Ex. 2.12.].

By construction, every polyhedral divisor $\mathcal{D}$ on a normal variety Y defines a sheaf $\mathcal{A}(\mathcal{D})$ of M-graded $\mathcal{O}_Y$-algebras and its ring $A(\mathcal{D})$ of global sections:

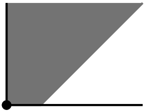
$$\mathcal{A}(\mathcal{D}) \ := \bigoplus_{u \in \omega \cap M} \mathcal{O}(\mathcal{D}(u)), \qquad A(\mathcal{D}) \ := \ \Gamma(Y(\mathcal{D}), \mathcal{A}(\mathcal{D})).$$

Now suppose that $\mathcal{D}$ is proper. Then [2, Thm. 3.1] guarantees that $A(\mathcal{D})$ is a normal affine algebra. Thus, we obtain an affine variety $X(\mathcal{D}) := \operatorname{Spec} A(\mathcal{D})$, which comes with an effective action of the torus $T = \operatorname{Spec} \mathbb{K}[M]$. By [2, Thm. 3.4], every normal affine variety with an effective torus action is isomorphic to some $X(\mathcal{D})$.

Example 4.2. Set $N = \mathbb{Z}^2$, let $\sigma \subseteq N_{\mathbb{Q}}$ be the cone generated by the vectors $(1, 1)$ and $(0, 1)$, and consider the σ-polyhedra Δ_0 and Δ_∞ given as follows:

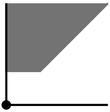


$\Delta_0 := (0, 1) + \sigma$

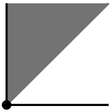


$\Delta_\infty := ([0, 1] \times 0) + \sigma$

Then we have a polyhedral divisor $\mathcal{D} := \Delta_0 \cdot \{0\} + \Delta_\infty \cdot \{\infty\}$ on $Y = \mathbb{P}_1$. Its degree $\deg(\mathcal{D})$ and tail cone $\operatorname{tail}(\mathcal{D})$ are given as



$\deg(\mathcal{D})$



$\operatorname{tail}(\mathcal{D})$

In particular, $\deg(\mathcal{D})$ is a proper subset of $\mathrm{tail}(\mathcal{D})$, and thus Remark 4.1 says that $\mathcal{D}$ is proper. The associated T-variety is $\mathbb{K}^3$ with the action

$$t \cdot z \quad = \quad (t_1^{-1} t_2 z_1, t_1 z_2, t_2 z_3).$$

As in the case of toric varieties, general T-varieties are obtained by gluing affine ones. In the combinatorial picture, the gluing leads to the concept of a divisorial fan, which we recall now. As before, let N be a finitely generated free abelian group, fix a pointed convex polyhedral cone $\sigma \in N_{\mathbb{Q}}$, and let Y be a normal variety. Consider two polyhedral divisors

$$\mathcal{D} \;=\; \sum_Z \Delta_Z \cdot Z, \qquad\qquad \mathcal{D}' = \sum_Z \Delta'_Z \cdot Z$$

both living on Y. The *intersection* of $\mathcal{D}$ and $\mathcal{D}'$ is the polyhedral divisor $\mathcal{D} \cap \mathcal{D}'$ on Y given by

$$\mathcal{D} \cap \mathcal{D}' \quad := \quad \sum_Z (\Delta'_Z \cap \Delta_Z) \cdot Z.$$

Moreover, given a (not necessarily closed) point $y \in Y$, we define the *slice* of $\mathcal{D}$ at y to be the polyhedron

$$\mathcal{D}_y \quad := \quad \sum_{y \in Z} \Delta_Z.$$

Note that the slice $\mathcal{D}_Y$ is the empty sum and hence equals the tail cone of $\mathcal{D}$. We say that $\mathcal{D}'$ is a *face* of $\mathcal{D}$ and write $\mathcal{D}' \preceq \mathcal{D}$ if $\mathcal{D}'_y \preceq \mathcal{D}_y$ holds for all $y \in Y$ and the T-equivariant morphism $X(\mathcal{D}') \to X(\mathcal{D})$ given by the inclusion $A(\mathcal{D}') \supseteq A(\mathcal{D})$ is an open embedding.

Remark 4.3. Suppose that in the above setting, we have $Y = \mathbb{P}_n$. As a consequence of [3, Lem. 6.7] the relation $\mathcal{D}' \preceq \mathcal{D}$ holds if and only if we have

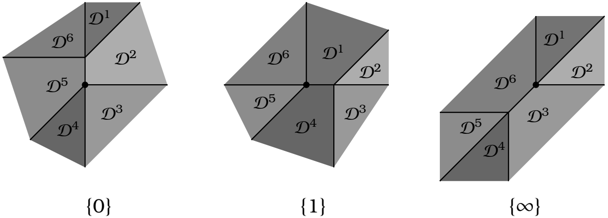
$$\mathcal{D}'_y \preceq \mathcal{D}_y \text{ for all } y \in Y, \qquad \deg(\mathcal{D}) \cap \mathrm{tail}(\mathcal{D}') \;=\; \deg(\mathcal{D}').$$

A *divisorial fan* is a finite set Ξ of polyhedral divisors such that for any two $\mathcal{D}, \mathcal{D}' \in \Xi$ we have $\mathcal{D} \succeq \mathcal{D}' \cap \mathcal{D} \preceq \mathcal{D}'$. For any $y \in Y$, we call the polyhedral complex Ξ_y defined by the slices $\mathcal{D}_y$ the *slice* of Ξ at y. We say that the divisorial fan Ξ is *complete* if Y is complete and each of its slices Ξ_y is a complete subdivision of $N_{\mathbb{Q}}$. The *locus* of Ξ is the open subset

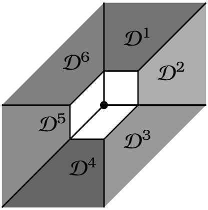
$$Y(\Xi) \;:=\; \bigcup_{\mathcal{D} \in \Xi} Y(\mathcal{D}) \;\subseteq\; Y.$$

Given a divisorial fan Ξ consisting of proper polyhedral divisors, [3, Thm. 5.3] guarantees that we can equivariantly glue the affine T-varieties $X(\mathcal{D})$ along the open subsets $X(\mathcal{D} \cap \mathcal{D}')$, where $\mathcal{D}, \mathcal{D}' \in \Xi$, to a T-prevariety $X(\Xi)$. If the divisorial fan Ξ is complete, then $X(\Xi)$ is a complete normal T-variety. By [3, Thm. 5.6], every normal variety with torus action is isomorphic to some $X(\Xi)$.

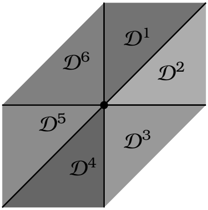
Example 4.4. Set $N := \mathbb{Z}^2$ and $Y := \mathbb{P}_1$. Consider the six polyhedral divisors $\mathcal{D}^1, \ldots, \mathcal{D}^6$ with coefficients over the points 0, 1 and ∞ as indicated below.



The collections of degrees $\deg(\mathcal{D}^i)$ and tail cones $\mathrm{tail}(\mathcal{D}^i)$ of these polyhedral divisors are given as



$\deg(\mathcal{D}^i)$



$\mathrm{tail}(\mathcal{D}^i)$

Remarks 4.1 and 4.3 yield that $\mathcal{D}^1, \ldots, \mathcal{D}^6$ are proper and form a divisorial fan Ξ. The T-variety $X(\Xi)$ is the projectivized cotangent bundle over $\mathbb{P}_2$.

For the description of the Cox ring of the T-variety X defined by a divisorial fan, we first of all need a description of the invariant prime divisors of X and their generic isotropy groups. For this, we introduce the following data.

Definition 4.5. Consider a divisorial fan Ξ on a normal projective variety Y, and let $Z \subseteq Y$ be a prime divisor.

(i) The *index* of a vertex $v \in \Xi_Z$ is the minimal positive integer $\mu(v)$ such that $\mu(v) \cdot v \in N$ holds.

(ii) We call a vertex $v \in \Xi_Z$ *extremal* if there is a $\mathcal{D} \in \Xi$ with $v \in \mathcal{D}_Z$ such that $\mathcal{O}(\mathcal{D}(u))$ is big on Z for any $u \in ((\mathcal{D}_Z - v)^\vee)^\circ$. The set of all extremal vertices $v \in \Xi_Z$ is denoted by $\Xi_Z^\times$

(iii) We call a ray $\varrho \in \Xi_Y$ *extremal* if there is a $\mathcal{D} \in \Xi$ with $\varrho \in \mathcal{D}_Y$ such that $\mathcal{O}(\mathcal{D}(u))$ is big on Y for any $u \in (\varrho^\perp \cap \omega)^\circ$. The set of all extremal rays $\varrho \in \Xi_Y$ is denoted by $\Xi_Y^\times$.

(iv) We say that the prime divisor Z is *irrelevant* if $\Xi_Z^\times$ is empty, and we denote by $Y^\circ \subseteq Y(\Xi)$ the open subset obtained by removing all irrelevant Z.

Remark 4.6. Let Ξ be a divisorial fan on $Y = \mathbb{P}_n$ and $Z \subseteq \mathbb{P}_n$ a prime divisor. Then every vertex $v \in \Xi_Z$ is extremal and a ray $\varrho \in \Xi_Y$ is extremal if and only if $\varrho \cap \deg(\mathcal{D}) = \emptyset$ holds for some $\mathcal{D} \in \Xi$ with $\varrho \in \mathcal{D}_Y$.

As shown in [20], the extremal vertices of Ξ are in bijection with the invariant prime divisors of $X = X(\Xi)$ intersecting X_0 and the extremal rays correspond to those contained in $X \setminus X_0$; see also Propositions 4.11

and 4.12. We will denote by D_{υ} the divisors given by extremal vertices $\upsilon \in \Xi_Z^{\times}$ and by E_{ϱ} those given by extremal rays $\varrho \in \Xi_Y^{\times}$. Then the divisor class group $\mathrm{Cl}(X)$ can be described as follows, see [20, Cor. 3.17].

Proposition 4.7. *Let Ξ be a divisorial fan on Y and set $X = X(\Xi)$. Then $\mathrm{Cl}(X)$ is generated by the classes $[D_{\upsilon}]$, $\upsilon \in \Xi_Z^{\times}$ and $[E_{\varrho}]$, $\varrho \in \Xi_Y^{\times}$ and the image of a canonical homomorphism $\mathrm{Cl}(Y^{\circ}) \to \mathrm{Cl}(X)$. The relations among these generators are*

$$\sum_{\upsilon \in \Xi_Z^{\times}} \mu(\upsilon) D_{\upsilon} = [Z], \quad \sum_{\varrho \in \Xi^{\times}} \langle u, v_{\varrho} \rangle E_{\varrho} + \sum_{Z} \sum_{\upsilon \in \Xi_Z^{\times}} \mu(\upsilon) \langle u, \upsilon \rangle D_{\upsilon} = 0,$$

where Z runs through the prime divisors of Y, u runs through (a basis of) the lattice M and $v_{\varrho} \in \varrho$ denotes the primitive lattice vector.

We are ready to compute the Cox ring of a T-variety $X = X(\Xi)$ in terms of its defining divisorial fan Ξ and the projective variety Y carrying Ξ. Let $Z_0, \ldots, Z_r \subseteq Y$ be the prime divisors having nontrivial slices $\Xi_{Z_0}, \ldots, \Xi_{Z_r}$.

Theorem 4.8. *There is a* $\mathrm{Cl}(X)$*-graded inclusion of Cox rings* $\mathcal{R}(Y^\circ) \to \mathcal{R}(X)$ *and a* $\mathrm{Cl}(X)$*-graded isomorphism*

$$\mathcal{R}(X) \quad \cong \quad \frac{\mathcal{R}(Y^\circ)[S_\varrho, T_\upsilon;\ \varrho \in \Xi_Y^\times, \upsilon \in \Xi_{Z_i}^\times,\ 0 \leq i \leq r]}{\langle T^{\mu_i} - 1_{Z_i};\ i = 0, \ldots, r\rangle},$$

where we set $T^{\mu_i} := \prod_{\upsilon \in \Xi_{Z_i}^\times} T_\upsilon^{\mu(\upsilon)}$ *and* $1_{Z_i} \in \mathcal{R}(Y^\circ)$ *denotes the canonical section of the prime divisor* $Z_i \subseteq Y$. *The grading is given by* $\deg T_\upsilon = [D_\upsilon]$ *and* $\deg S_\varrho = [E_\varrho]$.

As a direct consequence, we obtain the following description of the Cox ring of a T-variety of complexity one.

Corollary 4.9. *Let* Ξ *be a divisorial fan on* $Y = \mathbb{P}^1$ *having non-trivial slices* $\Xi_{a_0}, \ldots, \Xi_{a_r}$. *Then the Cox ring of* $X = X(\Xi)$ *is given by*

$$\frac{\mathbb{K}[S_\varrho, T_\upsilon;\ \varrho \in \Xi_Y^\times,\ \upsilon \in \Xi_{a_0}^\times \dot\cup \ldots \dot\cup \Xi_{a_r}^\times]}{\left\langle \sum_{i=0}^r \beta_i T^{\mu_i};\ \beta \in \mathrm{Rel}(\tilde{a}_0, \ldots, \tilde{a}_r)\right\rangle},$$

where $\widetilde{a}_i \in \mathbb{K}^2$ *represents* $a_i \in \mathbb{P}^1$, *we set* $T^{\mu_i} := \prod_{\upsilon \in \Xi_{a_i}^\times} T_\upsilon^{\mu(\upsilon)}$ *and* $\mathrm{Rel}(\widetilde{a}_0, \ldots, \widetilde{a}_r)$ *is a basis for the space of linear relations among* $\widetilde{a}_0, \ldots, \widetilde{a}_r$.

Note that an appropriate choice of a basis for the space of linear relations among $\widetilde{a}_0, \ldots, \widetilde{a}_r \in \mathbb{K}^2$ gives a trinomial representation of the Cox ring as in Theorem 1.3.

Example 4.10. Consider once more the divisorial fan Ξ and its associated variety $X(\Xi)$ of Example 4.4. According to Remark 4.6, there are no extremal rays and all six vertices

$$v_1, v_2 \in \Xi_{\{0\}}, \qquad v_3, v_4 \in \Xi_{\{1\}}, \qquad v_5, v_6 \in \Xi_{\{\infty\}}$$

are extremal, where we have $v_1 = v_3 = v_5 = 0 \in N$. Proposition 4.7 shows that $\mathrm{Cl}(X(\Xi))$ is freely generated by the classes of D_{v_1} and D_{v_2}. By Corollary 4.9, the Cox ring of $X(\Xi)$ is

$$\mathcal{R}(X(\Xi)) = \mathbb{K}[T_1, \ldots, T_6] \,/\, \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$$

with $\deg T_1 = \deg T_3 = \deg T_5 = [D_{v_1}]$ and $\deg T_2 = \deg T_4 = \deg T_6 = [D_{v_2}]$. Note that this presentation of the Cox ring shows that $X(\Xi)$ can be obtained as a $\mathbb{K}^*$-quotient of the Grassmannian $G(2, 4)$.

The rest of the section is devoted to proving Theorem 4.8, which basically means to express the input data of Theorem 1.2 in terms of polyhedral divisors. For this, we first have to recall further details of the construction of the T-variety $X(\Xi)$ associated to a divisorial fan Ξ on a projective variety Y. For every $\mathcal{D} \in \Xi$, we have the sheaf $\mathcal{A}(\mathcal{D})$ of normal M-graded $\mathcal{O}_Y$-algebras. Its relative spectrum $\widetilde{X}(\mathcal{D}) := \mathrm{Spec}_Y \mathcal{A}(\mathcal{D})$ comes with a T-action and we have canonical morphisms

$$\widetilde{X}(\mathcal{D}) \to Y, \qquad \widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$$

defined by $\mathcal{O}_Y \subseteq \mathcal{A}(\mathcal{D})$ and $A(\mathcal{D}) = \Gamma(\widetilde{X}(\mathcal{D}), \mathcal{O})$. The T-varieties glue together $\widetilde{X}(\mathcal{D})$ along the open subsets $\widetilde{X}(\mathcal{D} \cap \mathcal{D}')$ to a T-variety $\widetilde{X}(\Xi)$. These gluings are compatible with the above maps and one obtains a commutative diagram

$$\begin{array}{ccc} \widetilde{X}(\Xi) & \xrightarrow{\quad r \quad} & X(\Xi) \\ & {\scriptstyle\widetilde{\pi}} \searrow \quad \swarrow {\scriptstyle\pi} & \\ & Y & \end{array}$$

where $r\colon \widetilde{X}(\Xi) \to X(\Xi)$ is T-equivariant, birational and proper, $\widetilde{\pi}\colon \widetilde{X}(\Xi) \to Y$ is T-invariant and the rational map $\pi\colon X(\Xi) \to Y$ is defined in codimension two. Note that image of $\widetilde{\pi}$ is given by

$$\widetilde{\pi}(\widetilde{X}(\Xi)) = \bigcup_{\mathcal{D}\in\Xi} Y(\mathcal{D}) \subseteq Y.$$

The next step is a precise description of the T-invariant prime divisors $X(\Xi)$, see also [20, Prop. 3.13]. Consider an extremal vertex $v \in \Xi_Z^\times$ of $\mathcal{D} \in \Xi$, where $Z \subseteq Y(\mathcal{D})$ is a prime divisor. These data define a homogeneous ideal

$$I_v := \bigoplus_{u\in\mathcal{D}_Y^\vee\cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) \cap \{f \in K(Y);\ \mathrm{ord}_Z(f) > -\langle u, v\rangle\}$$

which turns out to be a prime ideal of height one. We define the corresponding prime divisor $D_v \subseteq X(\Xi)$ to be the closure of the zero set of I_v.

Proposition 4.11. *Set $X := X(\Xi)$. The assignment $v \to D_v$ induces a bijection between the extremal vertices of Ξ and the invariant prime divisors of X intersecting X_0. The extremal vertices of Ξ_Z correspond to the invariant prime divisors contained in $\overline{\pi^{-1}(Z)}$ and the generic isotropy group of D_v is cyclic of order $\mu(v)$.*

Proof. We may restrict to the affine case. Consider a proper polyhedral divisor $\mathcal{D}$, the corresponding sheaf of algebras $\mathcal{A} := \mathcal{A}(\mathcal{D})$ and its algebra of global sections $A := A(\mathcal{D})$. First we calculate the ideal of $\overline{\pi^{-1}(Z)} = r(\widetilde{\pi}^{-1}(Z))$. The inverse image ideal sheaf of $\mathcal{O}(-Z)$ is given by

$$\mathcal{O}(-Z) \cdot \mathcal{A} \quad = \quad \bigoplus_{u \in \mathcal{D}_Y^\vee \cap M} \mathcal{O}(\lfloor \mathcal{D}(u) \rfloor - Z).$$

The radical of the ideal $\Gamma(Y, \mathcal{O}(-Z) \cdot \mathcal{A}) \subseteq A$ is exactly the ideal we are looking for. It is given by book

$$\begin{aligned} I_Z \quad &= \bigoplus_{u \in \mathcal{D}_Y^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) \cap \{f;\ \mathrm{ord}_Z(f) > -\min\langle u, \mathcal{D}_Z \rangle\} \\ &= \bigcap_{v \in \mathcal{D}_Z} I_v. \end{aligned}$$

Note that we have $(I_v)_u = A_u$ if $\langle u, v \rangle \notin \mathbb{Z}$. Denote by $\kappa\colon \widetilde{Z} \to Z$ the normalization. If $\psi\colon \tilde{Y} \to Y$ is a desingularization, then we have $A(\mathcal{D}) = A(\psi^* \mathcal{D})$ and $I_v = \tilde{I}_v$, where $\tilde{I}_v$ is the ideal in $A(\psi^* \mathcal{D})$ corresponding to the vertex v in $(\psi^* \mathcal{D})_{f_*^{-1} Z}$. Hence, in the following we may assume that Y is smooth and thus every prime divisor is Cartier. Then, for the corresponding affine subschemes $V(I_v)$ we obtain the coordinate rings

$$\begin{aligned} A/I_v &= \bigoplus_{u \in (\mathcal{D}_Z - v)^\vee \cap M_v} \Gamma(Y, \mathcal{A}_u)/\Gamma(Y, \mathcal{A}_u(-Z)) \\ &\subseteq \bigoplus_{u \in (\mathcal{D}_Z - v)^\vee \cap M_v} \Gamma(Y, \mathcal{A}_u/\mathcal{A}_u(-Z)) \\ &\cong \bigoplus_{u \in (\mathcal{D}_Z - v)^\vee \cap M_v} \Gamma(Z, \mathcal{A}_u|_Z) \\ &\subseteq \bigoplus_{u \in (\mathcal{D}_Z - v)^\vee \cap M_v} \Gamma(\widetilde{Z}, \kappa^*(\mathcal{A}_u|_Z)) \\ &\cong A(\mathcal{D}_v). \end{aligned}$$

Here, we write $\mathcal{A}_u(-Z) := \mathcal{A}_u \otimes \mathcal{O}(-Z)$ as usual, $M_v \subseteq M$ is the sublattice consisting of all $u \in M$ with $\langle u, v \rangle \in \mathbb{Z}$ and $\mathcal{D}_v$ is a polyhedral divisor on Z with tail cone $\sigma_v := \mathbb{Q}_{\geq 0} \cdot (\mathcal{D}_Z - v)$ and lattice $M_v^* \supset N$ defined via the inclusion $\iota \colon Z \hookrightarrow Y$ as follows

$$\mathcal{D}_v := \sum_W (\mathcal{D}_W + \sigma_v) \cdot (\kappa \circ \iota)^* W.$$

Note that $(\kappa \circ \iota)^* Z$ is defined only up to linear equivalence as a divisor on $\widetilde{Z}$ but every choice will give isomorphic algebras $A(\mathcal{D}_v)$, compare [2, Cor. 8.9]. Our condition on the bigness of $\mathcal{D}(u)$ for $u \in \sigma_v^\vee$ implies that $\mathcal{D}_v$ is indeed proper for any extremal v. Hence,

$X(\mathcal{D}_v)$ is irreducible and of dimension $(\dim X - 1)$ in this case. If v is not extremal then $\mathcal{D}_v$ is the pullback of a proper polyhedral divisor on

$$Y_v \quad := \quad \operatorname{Proj} \bigoplus_k \Gamma(Y, \mathcal{O}(k \cdot \textstyle\sum_i \mathcal{D}_v(u_i)))$$

for some $u_i \in (\sigma_v^\vee)^\circ$. But Y_v is of smaller dimension than Y, since $\mathcal{O}(\mathcal{D}_v(u_i)) \cong \mathcal{O}(D(u_i))|_Z$ is not big. This implies, that $X(\mathcal{D}_v)$ is of dimension

$$\dim Y_v + \dim T \ < \ \dim Y + \dim T \ = \ \dim X - 1.$$

Since $\mathcal{D}(u)$ is semi-ample and $\mathcal{A}_u = \mathcal{O}(\mathcal{D}(u))$ holds, $\oplus_{k\geq 0} H^1(\mathcal{A}_{k\cdot u}(-Z))$ is finitely generated as a module over the ring $\oplus_{k\geq 0}\Gamma(Y, \mathcal{A}_{k\cdot u})$. The long exact cohomology sequence book

$$\begin{aligned} 0 \to & H^0(\mathcal{A}_{k\cdot u}(-Z)) \to H^0(\mathcal{A}_{k\cdot u}) \to \\ & \to H^0(\mathcal{A}_{k\cdot u}/(\mathcal{A}_{k\cdot u}(-Z)) \to H^1(\mathcal{A}_{k\cdot u}(-Z)) \to \dots \end{aligned}$$

shows that $\oplus_{k\geq 0}\Gamma(Y, \mathcal{A}_{k\cdot u}/\mathcal{A}_{k\cdot u}(-Z))$ is a finitely generated module over the ring $\oplus_{k\geq 0}\Gamma(Y, \mathcal{A}_{k\cdot u})/\Gamma(Y, \mathcal{A}_{k\cdot u}(-Z)$ The fact that $\Gamma(\widetilde{Z}, \kappa^*(\oplus_u \mathcal{A}_{k\cdot u}|_Z))$ is finitely generated over $\Gamma(Z, \oplus_u \mathcal{A}_{k\cdot u}|_Z)$ follows from the properties of the normalization map. Thus, $A(\mathcal{D}_v)$ is finitely generated over A/I_v and D_v is the image of $X(\mathcal{D}_v)$ under a finite morphism f, hence, D_v is irreducible and of codimension one; it is not hard to see that f even is the normalization map.

The fact that all homogeneous functions of weights $u \notin M_v$ vanish on D_v implies, that T does not act effectively on D_v but with generic isotropy group $M/M_v \cong \mathbb{Z}/\mu(v)\mathbb{Z}$. □

Now take an extremal ray $\varrho \in \Xi_Y^{\times}$ with $\varrho \in \mathcal{D}_Y$, where $\mathcal{D} \in \Xi$. Then define the associated invariant prime divisor E_ϱ of $X(\Xi)$ to be the closure of the zero set of $V(X(\mathcal{D}), I_\varrho)$, where I_ϱ is the homogeneous prime ideal of height one given by

$$I_\varrho \ := \bigoplus_{u \in \mathcal{D}_Y^\vee \setminus \varrho^\perp} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) \ \subseteq \ \Gamma(X(\mathcal{D}), \mathcal{O}).$$

Proposition 4.12. *Set $X := X(\Xi)$. The assignment $\varrho \to E_\varrho$ induces a bijection between the set of extremal rays of Ξ and the invariant prime divisors of X contained in $X \setminus X_0$.*

Proof. For a polyhedral divisor $\mathcal{D}$, the invariant prime divisors of $\widetilde{X}$ contained in $\widetilde{X}/\widetilde{X}_0$ correspond to the prime ideal sheaves given by not necessarily extremal rays $\rho \in \mathcal{D}_Y(1)$ as follows

$$\mathcal{I}_\rho \quad := \quad \bigoplus_{u \in \mathcal{D}_Y^\vee \setminus \rho^\perp} \mathcal{O}(\mathcal{D}(u)).$$

This can be seen locally. Consider an affine open subset $U \subset Y$ such that $\mathcal{D}|_U$ is trivial. Then $\widetilde{X}(\mathcal{D}|_U) \subseteq \widetilde{X}(\mathcal{D})$ is an open inclusion and we have

$$A(\mathcal{D}|_U) \quad = \quad \Gamma(U, \mathcal{O}_Y)[D_Y^\vee \cap M].$$

Now the claim follows from standard toric geometry, since the considered prime divisors correspond to ideals $I \subset A(\mathcal{D}|_U)$ with $I \cap \Gamma(U, \mathcal{O}_Y) = 0$.

The image under r corresponds to the ideal $I_v = \Gamma(\mathcal{I}_\rho)$ and for the coordinate ring of the corresponding subvariety we obtain

$$A(\mathcal{D})/I_v \;=\; \bigoplus_{u \in \rho^\perp \cap \mathcal{D}_Y^\vee \cap M} \Gamma(Y, \mathcal{D}(u)) \;=\; A(\mathcal{D}_\rho).$$

Here, $\mathcal{D}_\rho := \sum_Z p(\mathcal{D}_Z) \cdot Z$ is a polyhedral divisor on Y with tail cone $p(D_Y)$ and lattice $p(N)$, where p is the projection $N_\mathbb{Q} \to N_\mathbb{Q}/\mathbb{Q} \cdot \rho$.

The fact that ρ is extremal ensures that D_ρ is proper, which in turn implies that $V(I_\rho)$ has codimension one. □

Proof. Proof of Theorem 4.8 We first construct big open subsets $Y' \subseteq Y^\circ$ and $X' \subseteq X_0$. The set Y' is obtained by removing from Y° all the intersections $Z_i \cap Z_j$, where $0 \leq i < j \leq r$. To define X', denote by $\widetilde{E} \subseteq \widetilde{X}$ the exceptional locus of the contraction $r\colon \widetilde{X} \to X$ and set

$$\widetilde{X}' := (\pi^{-1}(Y') \cap \widetilde{X}_0) \setminus \widetilde{E} \subseteq \widetilde{X}_0, \qquad X' := r(\widetilde{X}') \subseteq X_0.$$

Then $\pi\colon \widetilde{X}' \to Y'$ is surjective and $r\colon \widetilde{X}' \to X'$ is an isomorphism. Moreover, the T-invariant map $\pi' := \pi \circ r^{-1}$ factors as

$$\begin{array}{ccc} X' & \xrightarrow{\quad \pi' \quad} & Y' \\ & \searrow \qquad \nearrow_{\varphi} & \\ & X'/T & \end{array}$$

Note that φ is birational and injective. Thus, φ is a local isomorphism and hence a separation for X'/T.

By Proposition 4.11, the prime divisors corresponding to the extremal vertices $v \in \Xi_{Z_i}^\times$ are precisely the irreducible components of the inverse image $(\pi')^{-1}(Z_i)$, and their generic T-isotropy is of order $\mu(v)$. Moreover, by Proposition 4.12, the prime divisors in $X \setminus X_0$

correspond to the extremal rays of Ξ. Now the assertion follows from Theorem 1.2. □

5. Applications and Examples

We first note some algebraic properties of the Cox ring of a variety with complexity one torus action. Recall the following concepts from [11, Def. 3.1]. Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ any K-graded integral $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$.

(i) We say that a nonzero nonunit $f \in R$ is *K-prime* if it is homogeneous and $f | gh$ with homogeneous $g, h \in R$ always implies $f | g$ or $f | h$.
(ii) We say that an ideal $\mathfrak{a} \subset R$ is *K-prime* if it is homogeneous and for any two homogeneous $f, g \in R$ with $fg \in \mathfrak{a}$ one has $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.
(iii) We say that a homogeneous prime ideal $\mathfrak{a} \subset R$ has *K-height d* if d is maximal admitting a chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_d = \mathfrak{a}$ of K-prime ideals.
(iv) We say that the ring R is *factorially graded* if every K-prime ideal of K-height one is principal.

Now, let X be a complete normal variety with finitely generated divisor class group and an algebraic torus action $T \times X \to X$ of complexity one. Then Theorem 1.3 provides a presentation of the Cox ring of X as book

$$\mathcal{R}(X) \quad \cong \quad \frac{\mathbb{K}[S_1, \ldots, S_m, T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle g_i;\ 0 \le i \le r-2 \rangle},$$

where the variables S_j and T_{ij} are homogeneous with respect to the $\mathrm{Cl}(X)$-grading and the relations g_i are $\mathrm{Cl}(X)$-homogeneous trinomials all having the same degree.

Proposition 5.1. *The Cox ring $\mathcal{R}(X)$ is factorially $\mathrm{Cl}(X)$-graded. In the presentation of Theorem 1.3, the generators S_k and T_{ij} define pairwise nonassociated $\mathrm{Cl}(X)$-prime elements and $\mathcal{R}(X)$ is a complete intersection.*

Proof. The fact that $\mathcal{R}(X)$ is factorially $\mathrm{Cl}(X)$-graded holds for any complete variety with a finitely generated Cox ring, use for example [11, Prop. 3.2]. Moreover, the variables S_k and T_{ij} define pairwise nonassociated $\mathrm{Cl}(X)$-prime elements, because their divisors are pairwise different H_X-prime divisors, where $H_X = \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$, use again [11, Prop. 3.2].

We show that $\mathcal{R}(X)$ is a complete intersection. This means to verify that $\mathcal{R}(X)$ is of dimension $m+n_0+\ldots+n_r-(r-1)$. Consider the torsor $\widehat{X} \to X$, and recall from the proof of Theorem 1.1, diagram 3.1, that we have quotients

$$\widehat{Y}_0 = \widehat{X}_0/G_0, \qquad \widehat{Z}_0 = \widehat{Y}_0/H_0, \qquad X_0/T \cong \widehat{Z}_0/F$$

where G_0 is an m-dimensional torus acting freely, H_0 is a finite group and F is a diagonalizable group acting freely and having the rank of $\mathrm{Cl}(X_0/T)$ as its dimension. In our situation, $X_0/T \cong \mathbb{P}_1(A, \mathfrak{n})$ is of dimension one and, by Proposition 2.3 has a divisor class group of rank $n_0 + \ldots + n_r - r$. Thus, the dimension of $\mathcal{R}(X)$ equals

$$\dim(\widehat{X}_0) = m+\dim(\widehat{Y}_0) = m+\dim(\widehat{Z}_0) = m+n_0+\ldots+n_r-r+1$$

□

We come to geometric applications of this observation. Note that each complete normal variety X with finitely generated divisor class group and a complexity one torus action is rational, because $X_0/T \cong \mathbb{P}_1(A, \mathfrak{n})$ is so. Thus, the varieties X in question are precisely the complete normal rational ones with a torus action of complexity one. If we impose additionally the condition that any two points of X admit a common affine neighbourhood, which holds e.g. for projective X, then Proposition 5.1 and [11, Thm. 4.19] ensure that X arises from a "bunched ring" $(R, \mathfrak{F}, \Phi)$, see [11, Def. 3.3, Constr. 3.4], where we may take $R = \mathcal{R}(X)$ and $\mathfrak{F} = (S_k, T_{ij})$. This allows us to apply the results provided in [11].

Corollary 5.2. *Let X be a complete normal rational variety with an effective algebraic torus action $T \times X \to X$ of complexity one and suppose that any two points of X admit a common affine neighbourhood. Then there exists a closed embedding $\imath\colon X \to X'$ into a toric variety X' with big torus $T' \subseteq X'$ such that*

(i) *$\imath\colon X \to X'$ is equivariant with respect to a T-action on X' given by a monomorphism $T \to T'$,*
(ii) *the image $\imath(X) \subseteq X'$ intersects T' and is a complete intersection of T-invariant hypersurfaces of X',*
(iii) *for every T'-invariant prime divisor $D' \subseteq X'$, the inverse image $\imath^{-1}(D') \subseteq X$ is a prime divisor,*
(iv) *$\imath\colon X \to X'$ defines a pullback isomorphism $\imath^*\colon \mathrm{Cl}(X') \to \mathrm{Cl}(X)$ on the level of divisor class groups.*

Proof. Apply the construction of a toric embedding given in [11, Constr. 3.13 and Prop. 3.14] to the defining bunched ring $(R, \mathfrak{F}, \Phi)$ of X, where $R = \mathcal{R}(X)$ and $\mathfrak{F} = (S_k, T_{ij})$, and use the fact that the S_k as well as the T_{ij} are homogeneous with respect to a lifting of the T-action to the torsor. □

Recall from the introduction that $E_k \subseteq X$ are the prime divisors supported in $X \setminus X_0$, that $D_{ij} \subseteq X$ are prime divisors intersecting X_0 and lying over a point $a_i \in X_0/T$ and l_{ij} is the order of the generic isotropy group of T along D_{ij}.

Corollary 5.3. *Let X be a complete normal rational variety with an effective algebraic torus action $T \times X \to X$ of complexity one.*

(i) *The cone of divisor classes without fixed components is given by*

$$\bigcap_{1 \le k \le m} \mathrm{cone}\big([E_s],[D_{ij}];\ s \neq k\big) \cap \bigcap_{\substack{0 \le i \le r \\ 1 \le j \le n_i}} \mathrm{cone}\big([E_k],[D_{st}];\ (s,t) \neq (i,j)\big).$$

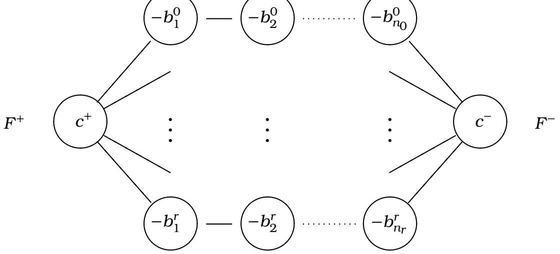
(ii) *For any $0 \le i \le r$, one obtains a canonical divisor for X by*

$$\max(0, r-1) \cdot \sum_{j=0}^{n_i} l_{ij} D_{ij} \;-\; \sum_{k=1}^{m} E_k \;-\; \sum_{i,j} D_{ij}.$$

Proof. By [4, Thm. 2.3], there is a small birational transformation $X \to X'$ with a projective X'. As X and X' share the same Cox ring, we may assume that X is projective. The assertions then follow from [11, Prop. 4.1 and Prop. 4.15]. □

Note that [20, Thm. 3.19] provides an equivalent description of the canonical divisor in terms of the defining divisorial fan.

The first non-trivial examples of torus actions of complexity one are $\mathbb{K}^*$-surfaces. Let us look at their Cox rings. Orlik and Wagreich associate in [17] to any smooth complete $\mathbb{K}^*$-surface X without elliptic fixed points a graph of the following shape:



The vertices of this graph represent certain invariant curves. The two (smooth) fixed point curves of X occur as F^+ and F^- in the graph. The other vertices represent the invariant irreducible contractible curves $D_{ij} \subseteq X$ different from F^+ and F^-. The label $-b^i_j$ is the self intersection number of D_{ij}, and two of the D_{ij} are joined by an edge if and only if they have a common (fixed) point. Every D_{ij} is the closure of a non-trivial $\mathbb{K}^*$-orbit.

We show how to read off the Cox ring from the Orlik-Wagreich graph. Suppose that X is rational. Then F^+ is rational as well and hence is a $\mathbb{P}_1$. Define l_{ij} to be the numerator of the canceled continued fraction

$$b_1^i - \cfrac{1}{b_2^i - \cfrac{1}{\cdots - \cfrac{1}{b_{j-1}^i}}}$$

Moreover, let a_i be the point in $F^+ \cap D_{i1}$ and write $a_i = [b_i, c_i]$ with $b_i, c_i \in \mathbb{K}$. Then, for every $0 \le i \le r$, set $k = j+1 = i+2$ and define a trinomial in $\mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i]$ as follows book

$$g_i \ := \ (c_k b_j - c_j b_k) f_i \ + \ (c_i b_k - c_k b_i) f_j \ + \ (c_j b_i - c_i b_j) f_k,$$

where $f_s := T_{s1}^{l_{s1}} \cdots T_{sn_s}^{l_{sn_s}}$.

Theorem 5.4. *Let X be a smooth complete rational $\mathbb{K}^*$-surface without elliptic fixed points. Then the assignments $S^\pm \mapsto 1_{F^\pm}$ and $T_{ij} \mapsto 1_{D_{ij}}$ define an isomorphism book*

$$\mathcal{R}(X) \quad \cong \quad \frac{\mathbb{K}[S^+, S^-, T_{ij};\ \ 0 \le i \le r,\ 1 \le j \le n_i]}{\langle g_i;\ 0 \le i \le r-2 \rangle}$$

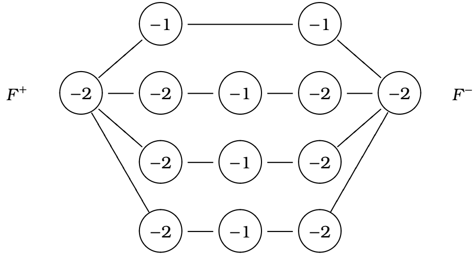
of $\mathrm{Cl}(X)$-graded rings, where the $\mathrm{Cl}(X)$-grading on the right hand side is defined by associating to $S^\pm$ the class of $F^\pm$ and to T_{ij} the class of D_{ij}.

Proof. The open set $X_0 \subseteq X$ is obtained by removing F^+, F^- and the isolated fixed points. By [17, Sec. 3.5], the number l_{ij} is the order of the isotropy group of the nontrivial $\mathbb{K}^*$-orbit in D_{ij}. Moreover, we have a canonical morphism $\pi\colon X_0/\mathbb{K}^* \to F^+$, with exceptional fibers $\pi^{-1}(a_i) = \{a_{i1}, \dots, a_{in_i}\}$, where a_{ij} represents the nontrivial $\mathbb{K}^*$-orbit of D_{ij}. Thus, the assertion follows from Theorem 1.3. □

For (possibly singular) $\mathbb{K}^*$-surfaces X with elliptic fixed points, the Cox ring can be computed as follows. Suitably resolving gives a $\mathbb{K}^*$-surface $\widetilde{X}$, called canonical resolution, where the elliptic fixed points are replaced with fixed point curves. Having computed the Cox ring $\mathcal{R}(\widetilde{X})$ as above, we easily obtain the Cox ring $\mathcal{R}(X)$. According to Theorem 1.3, we need the divisors of the type E_k and D_{ij} in X and the orders l_{ij} of the generic isotropy groups of the D_{ij}. Each of these divisors is the image of a non-exceptional divisor of the same type in $\widetilde{X}$; to see this for the D_{ij}, note that X_0 is the open subset of $\widetilde{X}_0$ obtained by removing the exceptional locus of $\widetilde{X} \to X$ and thus $X_0/\mathbb{K}^*$ is an open subset of $\widetilde{X}_0/\mathbb{K}^*$. Moreover, by equivariance, the orders l_{ij} in X are the same as in $\widetilde{X}$. Consequently, the Cox ring $\mathcal{R}(X)$ is obtained from $\mathcal{R}(\widetilde{X})$ by removing those generators that correspond to the exceptional curves arising from the resolution.

As the intersection graphs of their resolutions are known, see [1], the methods just outlined provide Cox rings of (possibly singular) Gorenstein del Pezzo $\mathbb{K}^*$-surfaces X; note that Derenthal computed in [8] the Cox rings of the minimal resolutions $\widetilde{X}$ without assuming existence of a $\mathbb{K}^*$-action for the cases that X is of degree at least 3 and $\mathcal{R}(\widetilde{X})$ is defined by a single relation. Moreover, the divisorial fans of Gorenstein del Pezzo $\mathbb{K}^*$-surfaces X are provided in [21], which allows us to use as well the approach via polyhedral divisors.

Example 5.5. We consider the family $X_{\mathfrak{a}}$ of Gorenstein Del Pezzo $\mathbb{K}^*$-surfaces over $\mathbb{K} \setminus \{0, 1\}$ of degree one and singularity type $2D_4$. The canonical resolution $\widetilde{X}_{\mathfrak{a}}$ of $X_{\mathfrak{a}}$ is obtained by minimally resolving the two singularities and, by [1, Thm. 8.3], its Orlik-Wagreich graph is given as

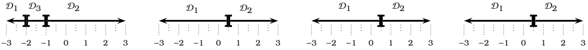


We have four points $a_0, \dots, a_3$, where $\{a_i\} = D_{i1} \cap F^+$. Note that the positions of these four points on $F^+ \cong \mathbb{P}_1$ may vary and the parameter λ is the cross ratio of a_0, a_1, a_2, a_3. The Cox rings of $\widetilde{X}_\lambda$ and X_λ are given by

$$\mathcal{R}(\widetilde{X}_\lambda) = \frac{\mathbb{K}[S_1, S_2, T_{01}, \dots, T_{33}]}{\left\langle \begin{array}{l} T_{01}T_{02}+T_{11}T_{12}^2T_{13}+T_{21}T_{22}^2T_{23}, \\ \lambda T_{11}T_{12}^2T_{13}+T_{21}T_{22}^2T_{23}+T_{31}T_{32}^2T_{33} \end{array} \right\rangle},$$

$$\mathcal{R}(X_\lambda) = \mathbb{K}[T_1, \dots, T_5] \Big/ \left\langle \begin{array}{l} T_1T_2+T_3^2+T_4^2, \\ \lambda T_3^2+T_4^2+T_5^2 \end{array} \right\rangle.$$

Now let us look at X_λ via its divisorial fan Ξ_λ. According to [21, Thm. 4.8], the divisorial fan Ξ_λ lives on $Y = \mathbb{P}_1$. Its non-trivial slices lie over the points $a_0, \dots, a_3 \in Y$ and are given in $N = \mathbb{Z}$ as follows:



Ξ_{a_0} Ξ_{a_1} Ξ_{a_2} Ξ_{a_3}

We compute the divisor class group $\mathrm{Cl}(X_{\mathfrak{D}})$. According to Remark 4.6, we have two extremal vertices v_1, v_2 in Ξ_{a_0} and one extremal vertex v_{i+2} in Ξ_{a_i} for $i = 1, 2, 3$. Let D_i be the prime divisor associated to v_i for $i = 1, \ldots, 5$ and denote by D_0 the positive generator of $\mathrm{Cl}(Y) = \mathbb{Z}$. Then Proposition 4.7 tells us that the divisor class group $\mathrm{Cl}(X_{\mathfrak{D}})$ is $\mathbb{Z}D_0 \oplus \ldots \oplus \mathbb{Z}D_5$ modulo the relations defined by the rows of the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & -2 & -1 & 1 & 1 & 1 \end{pmatrix}$$

The Smith Normal Form $S = U \cdot A \cdot V$ with unimodular transformation matrices U and V is given as

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

In particular, we conclude $\mathrm{Cl}(X_{\lambda}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Moreover, computing V^{-1} we see that the class of D_4 generates the free part and the classes of $D_3 - D_5$ and $D_4 - D_5$ generate the cyclic parts. Consulting Theorem 4.8 gives the Cox ring

$$\mathcal{R}(X_{\lambda}) = \mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2 + T_4^2,\ \lambda T_3^2 + T_4^2 + T_5^2 \rangle$$

whith the grading

$$\deg(T_1) = \deg(T_2) = (1, \overline{1}, \overline{0}), \qquad \deg(T_3) = (1, \overline{1}, \overline{1}),$$
$$\deg(T_4) = (1, \overline{0}, \overline{0}), \qquad \deg(T_5) = (1, \overline{0}, \overline{1}).$$

Proceeding as in this example, we are able to compute the Cox rings of all Gorenstein del Pezzo $\mathbb{K}^*$-surfaces and their minimal resolutions. Here comes the result for the cases of Picard number one and two.

Theorem 5.6. *Let X be a Gorenstein del Pezzo surface of Picard number at most two admitting a nontrivial $\mathbb{K}^*$-action. The following table provides the Cox rings of X and its minimal resolution $\widetilde{X}$ ordered by the degree* $\deg(X)$ *and the singularity type* $\mathrm{S}(X)$.

$\deg(X) = 1$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
$2D_4$	$\mathbb{K}[T_1,\dots,T_5]/\left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2, \\ \lambda T_3^2+T_4^2+T_5^2 \end{array} \right\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_{11}]/\left\langle \begin{array}{c} T_1T_2+T_6T_7T_3^2+T_8T_9T_4^2, \\ \lambda T_6T_7T_3^2+T_8T_9T_4^2+T_{10}T_{11}T_5^2 \end{array} \right\rangle$
E_6A_2	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^2T_2+T_3^3+T_4^3\rangle$	$\mathbb{K}[S,T_1,\dots,T_{11}]/\langle T_5T_1^2T_2+T_6T_7T_8^2T_3^3+T_9T_{10}T_{11}^2T_4^3\rangle$
E_7A_1	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^3T_2+T_3^4+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_{11}]/\langle T_5T_6^2T_1^3T_2+T_7T_8^2T_9^3T_{10}^3T_3^4+T_{11}T_4^2\rangle$
E_8	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^5T_2+T_3^3+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_{11}]/\langle T_5T_6^2T_7^3T_8^4T_1^5T_2+T_9T_{10}^2T_3^3+T_{11}T_4^2\rangle$

$\deg(X) = 2$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
$2A_3A_1$	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2+T_3^2+T_4^2\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_9]/\langle T_5T_1T_2+T_6T_7T_3^2+T_8T_9T_4^2\rangle$
A_5A_2	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2+T_3^3+T_4^3\rangle$	$\mathbb{K}[S,T_1,\dots,T_{10}]/\langle T_1T_2+T_5T_6T_7^2T_3^3+T_8T_9T_{10}^2T_4^3\rangle$
D_43A_1	$\mathbb{K}[S_1,T_1,T_2,T_3]/\langle T_1^2+T_2^2+T_3^2\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_9]/\langle T_4T_5T_1^2+T_6T_4T_2^2+T_8T_9T_3^2\rangle$
D_6A_1	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^2T_2+T_3^4+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_{10}]/\langle T_5T_1^2T_2+T_6T_7T_8^2T_9^3T_3^4+T_{10}T_4^2\rangle$
E_7	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^4T_2+T_3^3+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_{10}]/\langle T_5T_6^2T_7^3T_1^4T_2+T_8T_9^2T_3^3+T_{10}T_4^2\rangle$
$2A_3$	$\mathbb{K}[T_1,\dots,T_6]/\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2, \\ \lambda T_3T_4+T_5^2+T_6^2 \end{array} \right\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_{10}]/\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_7T_8T_5^2, \\ \lambda T_3T_4+T_7T_8T_5^2+T_9T_{10}T_6^2 \end{array} \right\rangle$
D_5A_1	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^3\rangle$	$\mathbb{K}[S,T_1,\dots,T_{10}]/\langle T_6T_1T_2^2+T_7T_3T_4^2+T_8T_9T_{10}^2T_5^3\rangle$
E_6	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_{10}]/\langle T_6T_7^2T_1T_2^3+T_8T_9^2T_3T_4^3+T_{10}T_5^2\rangle$

$\deg(X) = 3$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
A_5A_1	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2+T_3^4+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_9]/\langle T_1T_2+T_5T_6T_7^2T_8^3T_3^4+T_9T_4^2\rangle$
E_6	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^3T_2+T_3^3+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_9]/\langle T_5T_6^2T_1^3T_2+T_7T_8^2T_3^3+T_9T_4^2\rangle$
$2A_2A_1$	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_8]/\langle T_6T_1T_2+T_3T_4+T_7T_8T_5^2\rangle$
A_32A_1	$\mathbb{K}[S_1,T_1,\dots,T_4]/\langle T_1T_2+T_3^2+T_4^2\rangle$	$\mathbb{K}[S_1,S_2,T_1,\dots,T_8]/\langle T_1T_2+T_5T_6T_3^2+T_7T_8T_4^2\rangle$
A_4A_1	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4+T_5^3\rangle$	$\mathbb{K}[S,T_1,\dots,T_9]/\langle T_6T_1T_2^2+T_3T_4+T_7T_8T_9^2T_5^3\rangle$
D_5	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^2+T_5^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_9]/\langle T_6T_7^2T_1T_2^3+T_8T_3T_4^2+T_9T_5^2\rangle$

$\deg(X) = 4$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
D_5	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1^2T_2+T_3^3+T_4^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_8]/\langle T_5T_1^2T_2+T_6T_7^2T_3^3+T_8T_4^2\rangle$
A_3A_1	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^3\rangle$	$\mathbb{K}[S,T_1,\dots,T_8]/\langle T_1T_2+T_3T_4+T_6T_7T_8^2T_5^3\rangle$
A_4	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_8]/\langle T_6T_7^2T_1T_2^3+T_3T_4+T_8T_5^2\rangle$
D_4	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^2\rangle$	$\mathbb{K}[S,T_1,\dots,T_8]/\langle T_6T_1T_2^2+T_7T_3T_4^2+T_8T_5^2\rangle$

$\deg(X) = 5$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
A_3	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2+T_3T_4+T_5^2\rangle$	$\mathbb{K}[S,T_1,\ldots,T_7]/\langle T_6T_1T_2^2+T_3T_4+T_7T_5^2\rangle$
A_4	$\mathbb{K}[T_1,\ldots,T_4]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{K}[S,T_1,\ldots,T_7]/\langle T_1T_2+T_5T_6^2T_3^3+T_7T_4^2\rangle$

$\deg(X) = 6$

S(X)	$\mathcal{R}(X)$	$\mathcal{R}(\widetilde{X})$
A_2	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle$	$\mathbb{K}[S,T_1,\ldots,T_6]/\langle T_1T_2+T_3T_4+T_6T_5^2\rangle$

Finally, we consider equivariant vector bundles over a toric variety X arising from a fan Σ and ask for the Cox rings of their projectivizations. We will use Klyachko's description [13] of equivariant reflexive sheaves over X; we will follow Perling's notation [19] in terms of families of complete *increasing* filtrations.

We recall the basic constructions. Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank r on X. Then $\mathcal{E}$ is trivial over the big torus $T \subseteq X$. Moreover, for every ray $\rho \in \Sigma^{(1)}$, the sheaf $\mathcal{E}$ splits over the affine chart $X_\rho \subseteq X$ and hence is even trivial there. This gives us identifications

$$\Gamma(X_\varrho, \mathcal{E}) \ \subseteq \ \Gamma(T, \mathcal{E}) \ = \ E \otimes \Gamma(T, \mathcal{O}_X) \ = \ E \otimes \mathbb{K}[M]$$

with an r-dimensional vector space E. Fix generators $e_{\varrho,1} \otimes \chi^{u_{\varrho,1}}, \ldots, e_{\varrho,r} \otimes \chi^{u_{\varrho,r}}$ for every $\Gamma(X_\varrho, \mathcal{E})$. Then $\mathcal{E}$ is determined by the family of complete increasing filtrations $E^\varrho(i)$, where $\varrho \in \Sigma^{(1)}$, of E defined by

$$E^\varrho(i) \quad := \quad \operatorname{lin}(e_{\varrho,j};\ \langle u_{\varrho,j}, v_\varrho \rangle \leq i),$$

where $v_\varrho \in \varrho$ denotes the primitive lattice vector. Conversely, given any family of complete increasing filtrations $E^\varrho(i)$, where $\varrho \in \Sigma^{(1)}$, of $E = \mathbb{K}^r$, one obtains an equivariant reflexive sheaf $\mathcal{E}$ of rank r over X by defining its sections over the affine charts $X_\sigma \subseteq X$, where $\sigma \in \Sigma$, to be

$$\Gamma(X_\sigma, \mathcal{E}) \ := \ \bigoplus_{u \in M} \left(\bigcap_{\varrho \in \sigma^{(1)}} E^\varrho(\langle u, v_\varrho \rangle) \right) \otimes \chi^u \ \subseteq \ E \otimes \mathbb{K}[M].$$

In our first result, we compute the Cox ring of the projectivization $\mathbb{P}(\mathcal{E})$, see [12, p. 162], of a locally free sheaf $\mathcal{E}$ of rank two over a complete toric variety X arising from a fan Σ. Let $E^\varrho(i)$, where $\varrho \in \Sigma^{(1)}$, be the family of filtrations describing $\mathcal{E}$, let $\mathcal{L}$ be the set of one-dimensional subspaces of E occurring in these filtrations, for every $L \in \mathcal{L}$ fix a generator e_L, and denote by $\mathrm{Rel}(\mathcal{L})$ the space of linear relations among the e_L. Moreover, let i_k^ϱ be the smallest integer such that $\dim E^\varrho(i_k^\varrho) > k$ and set $L^\varrho := E^\varrho(i_0^\varrho)$.

Theorem 5.7. *Let $\mathcal{E}$ be an equivariant locally free sheaf of rank two over a complete toric variety X defined by a fan Σ. Then the Cox ring of the projectivization $\mathbb{P}(\mathcal{E})$ is given as book*

$$\mathcal{R}(\mathbb{P}(\mathcal{E})) = \frac{\mathbb{K}[S_\varrho, T_L;\ \varrho \in \Sigma^{(1)},\ L \in \mathcal{L}]}{\langle \sum_{L \in \mathcal{L}} \mathfrak{H}_L S^L T_L;\ \mathfrak{H} \in \mathrm{Rel}\,(\mathcal{L}) \rangle},$$

where $S^L := \prod_{\varrho,\ L^\varrho = L} S_\varrho^{i_1^\varrho - i_0^\varrho}$.

Example 5.8. Let $\mathcal{T}$ be the sheaf of sections of the tangent bundle of the projective plane $\mathbb{P}_2$; then $\mathbb{P}(\mathcal{T})$ is the projectivized cotangent bundle. As a toric variety, $\mathbb{P}_2$ is given by the complete fan in $\mathbb{Q}^2$ with the rays

$$\varrho_1 = \mathbb{Q}_{\geq 0} \cdot e_1, \qquad \varrho_2 = \mathbb{Q}_{\geq 0} \cdot e_2, \qquad \varrho_0 = \mathbb{Q}_{\geq 0} \cdot e_0,$$

where $e_1, e_2 \in \mathbb{Q}^2$ are the canonical basis vectors and we set $e_0 := -e_1 - e_2$. The filtrations of the tangent sheaf are given as

$$E^{\rho}(i) = \begin{cases} 0, & i < -1, \\ \mathbb{K}\cdot\rho, & i = -1, \\ E, & i > -1. \end{cases}$$

As generators for the one-dimensional subspaces we may choose $e_1, e_2, e_0 \in \mathbb{K}^2$. The linear relations between them are spanned by $(1, 1, 1) \in \mathbb{K}^3$. Hence, as in Example 4.10, we obtain

$$\mathcal{R}(\mathbb{P}(\mathcal{T})) = \mathbb{K}[S_1, S_2, S_3, T_1, T_2, T_3] \,/\, \langle T_1S_1+T_2S_2+T_3S_3\rangle.$$

More generally, we may calculate the Cox ring of the projectivized cotangent bundle on an arbitrary smooth complete toric variety X arising from a fan Σ. We distinguish two types of rays $\rho \in \Sigma^{(1)}$: those with $-\rho \notin \Sigma^{(1)}$ and those with $-\rho \in \Sigma^{(1)}$. Denote by $\mathcal{L}$ the set containing all rays of the first type and one representative for every pair of the second type. Moreover, let $\mathrm{Rel}(\mathcal{L})$ denote the tuples $\mathfrak{a} \in \mathbb{K}^{\mathcal{L}}$ such that $\sum_{\rho\in\mathcal{L}} \mathfrak{a}_\rho v_\rho = 0$, where $v_\rho \in \rho$ denote the primitive generator.

Theorem 5.9. *Let X be a smooth complete toric variety arising from a fan Σ, and denote by $\mathcal{T}_X$ the sheaf* *of sections of the tangent bundle over X. Then the Cox ring of the projectivization $\mathbb{P}(\mathcal{T}_X)$ is given by*

$$\mathcal{R}(\mathbb{P}(\mathcal{T}_X)) = \frac{\mathbb{K}[S_\varrho, T_\tau;\ \varrho \in \Sigma^{(1)},\ \tau \in \mathcal{L}]}{\left\langle \sum_{\varrho \in \mathcal{L}} \mathfrak{n}_\varrho S^\varrho T_\varrho;\ \mathfrak{n} \in \mathrm{Rel}\,(\mathcal{L}) \right\rangle},$$

where $S^\varrho := \begin{cases} S_\varrho S_{-\varrho} & -\varrho \in \Sigma^{(1)}, \\ S_\varrho & \text{else.} \end{cases}$

Remark 5.10. Let Σ be a fan in a lattice N having rays $\varrho_1, \ldots, \varrho_s$ as its maximal cones, Z the associated toric variety and $T \subseteq Z$ the acting torus. Given a primitive sublattice $L \subseteq N$, consider the action of the corresponding subtorus $H \subseteq T$ on Z. Let $P \colon N \to N' := N/L$ denote the projection. The generic isotropy group $H_\varrho \subseteq H$ along the toric divisor $D_\varrho \subseteq Z$ corresponding to a ray $\varrho \in \Sigma$ is one-dimensional if $P(\varrho) = 0$ holds and finite otherwise; in the latter case it is given by

$$\mathbb{X}(H_\varrho) \quad = \quad (\mathrm{lin}(P(\varrho)) \cap N') \ / \ P(\mathrm{lin}(\varrho) \cap N).$$

In particular, the set $Z_0 \subseteq Z$ is the toric subvariety corresponding to the subfan $\Sigma_0 \subseteq \Sigma$ obtained by removing all ϱ with $P(\varrho) = \{0\}$. For an affine chart

$Z_\varrho \subseteq Z_0$, the orbit space Z_ϱ/H is the affine toric variety $Z'_{P(\varrho)}$ corresponding to the ray $P(\varrho)$ in N'. Gluing these $Z'_{P(\varrho)}$ along their common big torus T/H gives the toric prevariety Z_0/H.

There is a canonical separation $\pi\colon Z_0/H \to Z'$, where Z' is the toric variety defined by the fan $P(\Sigma_0)$ in N' having $\{P(\varrho);\ \varrho \in \Sigma_0\}$ as its set of maximal cones. Note that the inverse image $\pi^{-1}(D_{P(\varrho)})$ of the divisor $D_{P(\varrho)} \subseteq Z'$ corresponding to $P(\varrho) \in P(\Sigma_0)$ is the disjoint union of all divisors $D_\tau \subseteq Z_0$ with $P(\tau) = P(\varrho)$.

Proof. Proof of Theorem 5.7 and Theorem 5.9 In order to use Theorem 1.2, we have to study the map $\pi \circ q$ obtained by composing the quotient $q\colon \mathbb{P}(\mathcal{E})_0 \to \mathbb{P}(\mathcal{E})_0/T$ with the separation $\pi\colon \mathbb{P}(\mathcal{E})_0/T \to Y$. This done in three steps. First we cover $\mathbb{P}(\mathcal{E})$ by affine toric charts and describe the quotient map on these charts using Remark 5.10. Then we collect the data for Theorem 1.2 in every chart. In the last step we will see how this local data fit into the global picture.

Step 1: the toric charts. We may assume that the maximal cones of Σ are just the rays $\varrho \in \Sigma$. On an affine chart X_ϱ any equivariant locally free sheaf $\mathcal{E}$ is actually free with homogeneous generators $s_{\varrho,0}, \ldots, s_{\varrho,r}$

of the form $s_{\varrho,i} = e_{\varrho,i} \otimes \chi^{u_{\varrho,i}}$. Choosing an appropriate order, we may achieve $\langle u_{\varrho,k}, v_\varrho \rangle = i_k$. Then $\mathbb{P}(\mathcal{E}|_{X_\varrho})$ is given as

$$\mathbb{P}(\mathcal{E}|_{X_\varrho}) = \operatorname{Proj}_{X_\varrho}(S(\mathcal{E}|_{X_\varrho})) = \operatorname{Proj} \mathbb{K}[\varrho^\vee \cap M][s_{\varrho,0}, \dots, s_{\varrho,r}],$$

where $\deg(s_{\varrho,i}) := 1$. So, $\mathbb{P}(\mathcal{E}|_{X_\varrho})$ is $X_\varrho \times \mathbb{P}^r$ but endowed with a special T^n-action. This action can be extended to an T^{n+r}-action by assigning to s_i the weight $(u_i, b_i) \in M \times M'$ for $i = 0, \dots r$. Here, $M' \cong \mathbb{Z}^r$ and $b_1, \dots, b_r$ is a basis and $b_0 := 0$. As a consequence we can describe $\mathbb{P}(\mathcal{E}|_{X_\varrho})$ as a toric variety and the endowed T^n-action by an inclusion $T^n \hookrightarrow T^{n+r}$, which corresponds to the lattice inclusion $N \hookrightarrow N \times N'$, where $N' := (M')^*$.

We describe the corresponding fan in $N \times N'$. Denote by $b_1^*, \dots, b_r^*$ the dual basis in N' and set $b_0^* := -\sum b_i^*$. Moreover, set

$$\varrho_i := \tilde{\varrho} + \operatorname{cone}(b_0^*, \dots, b_{i-1}^*, b_{i+1}^*, \dots, b_r^*),$$
$$\tilde{\varrho} := \mathbb{Q}_{\geq 0} \cdot (v_\varrho, -\sum_{k=0}^{r} i_k^\varrho b_k^*).$$

Then $\varrho_0, \dots, \varrho_r$ are the maximal cones of the fan we are looking for. Indeed, cover $\mathbb{P}(\mathcal{E}|_{X_\varrho})$ by the affine charts $\operatorname{Spec} \mathbb{K}[\rho^\vee \cap M][\frac{s_0}{s_i}, \ldots, \frac{s_r}{s_i}]$. Then

$$\begin{array}{rcl} \mathbb{K}[\rho^\vee \cap M][\frac{s_0}{s_i}, \ldots, \frac{s_r}{s_i}] & \to & \mathbb{K}[\rho_i^\vee \cap M \times M'], \\ \dfrac{s_j}{s_i} & \mapsto & \chi^{(u_j - u_i, b_j - b_i)}, \\ \chi^u & \mapsto & \chi^{(u,0)} \end{array}$$

defines an equivariant isomorphism from $\mathbb{P}(\mathcal{E}|_{X_\rho})$ with the extended torus action onto the toric variety arising from the fan just defined, see also [16, pp. 58-59].

Step 2: local quotient maps. In the setting of Remark 5.10, the map $P\colon N \times N' \to N'$ is the projection to the second factor and we deduce that the separation of $\mathbb{P}(\mathcal{E}|_{X_\rho})_0/T$ has the fan Σ' consisting of the rays $\mathbb{Q}_{\geq 0} \cdot b_0^*, \ldots, \mathbb{Q}_{\geq 0} \cdot b_r^*, P(\tilde{\rho})$ and the trivial cone.

If $\mathcal{E}$ is an equvariant locally free sheaf of rank two, then we obtain $\tilde{\rho} = \mathbb{Q}_{\geq 0} \cdot (v_\rho, (i_1 - i_0)b_0)$ and Σ' is the unique fan Δ of $\mathbb{P}^1$. If $\mathcal{E}$ is the sheaf of sections of the tangent bundle of X, then we have $E = N \otimes \mathbb{K}$ and the filtrations

$$E^\rho(i) = \begin{cases} 0, & i < -1, \\ \mathbb{K} \cdot \rho, & i = -1, \\ E, & i > -1. \end{cases}$$

Thus, by the chosen order of the $s_{\varrho,i}$, we obtain $\tilde{\varrho} = \mathbb{Q}_{\geq 0} \cdot (v_\varrho, b_0)$ and $\Sigma' = \{0, \mathbb{Q}_{\geq 0} \cdot b_0^*, \ldots, \mathbb{Q}_{\geq 0} \cdot b_r^*\}$. Hence, Σ' is a subfan of a fan Δ with $X_\Delta \cong \mathbb{P}^r$ and we have a rational toric map $p_\varrho : \mathbb{P}(\mathcal{E}|_{X_\varrho}) \dashrightarrow \mathbb{P}^r$, which is defined on a big open subset.

Now we locally collect the data for Theorem 1.2 using Remark 5.10. In the preimage $P^{-1}(\mathbb{Q}_{\geq 0} \cdot b_0)$, we find the rays $\tau := \mathbb{Q}_{\geq 0} \cdot (0, b_0)$ and $\tilde{\varrho}$. Hence, the prime divisor in $Y = X_{\Xi'}$ corresponding to $\mathbb{Q}_{\geq 0} \cdot b_0$ has two invariant prime divisors in its preimage under the map $\pi \circ q$.

The lattice elements of τ are mapped onto the lattice elements of $\mathbb{Q}_{\geq 0} \cdot b_0$, hence T acts effectively on the corresponding prime divisor. The lattice generated by $P(\tilde{\varrho} \cap (N \times N'))$ has index $i_1^\varrho - i_0^\varrho$ in $\mathbb{Z} \cdot b_0$. This implies that the corresponding prime divisor has a generic isotropy group of order $i_1 - i_0$ (which is equal to 1 in the case of $\mathcal{T}_X$).

Step 3: the global picture. We identify $\mathbb{P}(E^*)$ with X_Δ via the isomorphism φ_ϱ induced by the following homomorphism of the homogeneous coordinate rings

$$S(E) \to \mathbb{K}[\chi^{b_0}, \ldots, \chi^{b_r}], \qquad e_i \mapsto \chi^{b_i}.$$

Note that the map $\varphi_\varrho \circ p_\varrho$ (considered as a rational map $\mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(E^*)$) no longer depends on the choice of ϱ and $s_{\varrho,0}, \ldots, s_{\varrho,r}$, because over $\mathbb{P}(\mathcal{E}|_T)$ it is just the projection $\mathbb{P}(\mathcal{E}|_T) = \mathbb{P}(E^*) \times T \to \mathbb{P}(E^*)$ given by

$$\begin{array}{ccccc} S(E) & \hookrightarrow & \mathbb{K}[M][s_{\varrho,0}, \ldots, s_{\varrho,r}] & = & \Gamma(T, S(\mathcal{E})) \\ e_i & \mapsto & \chi^{-u_i} \cdot s_{\varrho,i} & = & e_i \otimes \chi^0 \end{array}.$$

Note that φ_ϱ maps the prime divisor corresponding to the ray $\mathbb{Q}_{\geq 0} \cdot b_0$ onto $E^\varrho(i_0)^\perp = e_{\varrho,0}^\perp \subset \mathbb{P}(E^*)$. Putting things together we obtain

- A ray ϱ with $i_0^\varrho = i_1^\varrho$ corresponding to a divisor outside of $\mathbb{P}(\mathcal{E})_0$.
- Since $(\pi \circ q)|_{\mathbb{P}(\mathcal{E}|_T)}$ is the equivariant projection $\mathbb{P}(\mathcal{E}|_T) = \mathbb{P}(E^*) \times T \to \mathbb{P}(E^*)$ the closure $\overline{(\pi \circ q)^{-1}(Z) \cap \mathbb{P}(\mathcal{E}|_T)}$ is always a prime divisor with effective T-action.
- If $i_0^\varrho < i_1^\varrho$ the ray ϱ corresponds to an additional invariant prime divisor in the preimage of $E^\varrho(i_0)^\perp$ with generic isotropy group of order $i_1 - i_0$.

Inspecting the filtrations for $\mathcal{T}_X$, we see that $i_1^\varrho - i_0^\varrho = 1$ for every ray ϱ and $E^\varrho(i_0^\varrho) = E^\tau(i_0^\tau)$ if and only if $\tau = \pm\varrho$. Using Theorem 1.2, we obtain the desired results. $\square$

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