Autumn Semester 2016–2017

MATH41072/MATH61072 *Algebraic topology*

§4. Simplicial Complexes and Simplicial Homology

Geometric simplicial complexes

4.1 Definition. A finite subset $\{ v_0, v_1, \ldots, v_r \} \subset \mathbb{R}^n$ is said to be *affinely independent* if the set of vectors $\{ v_1 - v_0, v_2 - v_0, \ldots, v_r - v_0 \}$ is linearly independent. By convention, when $r = 0$ we say that the singleton $\{v_0\}$ is affinely independent.
Given an affinely independent set $\{ v_0, v_1, \ldots, v_r \} \subset \mathbb{R}^n$ then the *closed r-simplex* with *vertices* v_0, v_1, $\ldots$, v_r is the subset

$$\langle v_0, v_1, \ldots, v_r \rangle = \Big\{ \sum_{i=0}^{r} t_i v_i \;\Big|\; t_i \geqslant 0,\ \sum_{i=0}^{r} t_i = 1 \Big\} \subset \mathbb{R}^n.$$

The *interior* of the simplex $\sigma = \langle v_0, v_1, \ldots, v_r \rangle$ is the set of points of the simplex for which all $t_i > 0$, it is denoted by σ°. The *boundary* of the simplex consists of all other points of the simplex and it is often denoted by $\partial\sigma$.
The *dimension* of the simplex is r.
If $\sigma = \langle v_0, v_1, \ldots, v_r \rangle$ is an r-simplex, then a *face* of σ is a (non-empty) simplex τ whose vertices are a subset of the vertices of σ. We write $\tau \prec \sigma$ to indicate that τ is a face of σ. A *proper face* of σ is a face other than σ itself.

Note. The plural of 'simplex' is 'simplices'.

4.2 Definition. The *standard r-simplex* is given by

$$\Delta^r = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r+1} \rangle = \Big\{ (t_0, t_1, \ldots t_r) \;\Big|\; t_i \geqslant 0,\ \sum_{i=0}^{r} t_i = 1 \Big\} \subset \mathbb{R}^{r+1}$$

where $\{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r+1} \}$ is the standard basis of $\mathbb{R}^{r+1}$ given by the columns of the $(r+1) \times (r+1)$ identity matrix.

4.3 Definition. A *(geometric) simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(a) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$,

(b) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

We write $V(K)$ for the set of vertices of K.

4.4 Definition. The *dimension* of a simplicial complex K, denoted $\dim K$, is the largest dimension of a simplex in K.

4.5 Definition. The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology.

4.6 Definition. Given a simplicial complex K and a non-negative integer r, the *r-skeleton* of K, denoted $K^{[r]}$ is the set of simplices in K of dimension no greater than r. Clearly, the r-skeleton of K is a simplicial complex.

4.7 Definition. A *triangulation* of a topological space X is a homeomorphism $h\colon |K| \to X$ from the underlying space of some simplicial complex K. A topological space which has a triangulation is said to be *triangulable.*

4.8 Examples. (a) A simplicial surface K as defined in Definition 2.3 determines a simplicial complex $\bar{K}$ of dimension 2. The simplicial complex $\bar{K}$ consists of the triangles of simplicial surface K together with the edges of K and the vertices of K. A 2-dimensional simplicial complex is much more general than a simplicial surface since we do not require the connectivity condition or the link condition.

(b) For a non-negative integer n, the n-simplex together with all of its faces is a simplicial complex which we denote $\bar{\Delta}^n$. It is intuitively clear that $|\bar{\Delta}^n|$ is homeomorphic to the n-ball D^n (by radial projection) and so the n-ball is a triangulable. [A formal proof of this is indicated on Problems 4.]

(c) Similarly, for $n \geqslant 1$, the underlying space of $(\bar{\Delta}^n)^{[n-1]}$, the $(n-1)-$*skeleton* of $\bar{\Delta}^n$, is homeomorphic to the $(n-1)$-sphere S^{n-1} which is therefore triangulable. [This is a generalization of Example 2.8(a).]

Abstract simplicial complexes

4.9 Proposition. Each point of an r-simplex $\sigma = \langle v_0, v_1, \ldots, v_r \rangle$ may be uniquely written in the form $x = \sum_{i=0}^{r} t_i v_i$, with $\sum t_i = 1$. The coefficients t_0, t_1, $\ldots t_r$ are called the *barycentric coordinates* of the point x.

Proof. Exercise. □

4.10 Definition. Given two simplicial complexes K_1 and K_2, an *isomorphism* $f\colon K_1 \to K_2$ is given by a bijection $f\colon V(K_1) \to V(K_2)$ such that $\langle v_0, v_1, \ldots, v_r \rangle$ is an r-simplex of K_1 if and only if $\langle f(v_0), f(v_1), \ldots, f(v_r) \rangle$ is a r-simplex of K_2.

4.11 Corollary. Given two r-simplices $\sigma = \langle v_0, v_1, \ldots, v_r\rangle$ and $\sigma' = \langle v_0', v_1', \ldots, v_r'\rangle$ then a homeomorphism $\sigma \to \sigma'$ may be defined by $\sum_{i=0}^{r} t_i v_i \mapsto \sum_{i=0}^{r} t_i v_i'$.
More generally, given an isomorphism $f\colon K_1 \to K_2$ of simplicial complexes, then f induces a homeomorphism $|f|\colon |K_1| \to |K_2|$ of the underlying spaces by

$$|f|\Big(\sum_{i=0}^{r} t_i v_i\Big) = \sum_{i=0}^{r} t_i v_i'.$$

Proof. Exercise. □

4.12 Definition. An *(abstract) simplicial complex* L consists of a finite set $V = V(L)$ (the *vertex set* of L) together with a set $S = S(L)$ of subsets of V (the *simplex set* of L) such that

(a) **the vertex condition:** $v \in V \Rightarrow \{v\} \in S$;

(b) **the face condition:** $\sigma \in S$ and $\tau \subset \sigma \Rightarrow \tau \in S$.

An element $v \in V$ is called a *vertex* of L. An element $\sigma \in S$ is called a *simplex* of L. Given simplices σ and τ, if $\tau \subset \sigma$ then we say that τ is a *face* of σ. The *dimension* of $\sigma = \{\, v_0, v_1, \ldots, v_r\}$ is r. The dimension of K is the largest dimension of a simplex in K.
It is clear that a geometric simplicial complex K determines an abstract simplicial complex L with vertex set $V(L) = V(K)$ and simplex set $S(L)$ consisting of the subsets of of the vertex set which are the vertices of a simplex in K.
Conversely, a *realization* of an abstract simplicial complex L is a geometric simplicial complex K with a bijection $f\colon V(L) \to V(K)$ such that $\{v_0, v_1, \ldots, v_r\} \in S(L)$ if and only if $\langle f(v_0), f(v_1), \ldots, f(v_r)\rangle \in K$.

4.13 Proposition. Each abstract simplicial complex L has a realization K and given two realizations K_1 and K_2 then the underlying spaces $|K_1|$ and $|K_2|$ are homeomorphic. Hence, we can define the underlying space of an abstract simplicial complex L by $|L| = |K|$ where K is a geometric realization of L and this is well-defined up to homeomorphism.

Proof. If L has n vertices then we find a geometric realization of L as a geometric simplicial complex in $\mathbb{R}^n$ with vertices given by the standard basis vectors of $\mathbb{R}^n$ (the argument of Example 2.8(b) deals with the intersection condition). The homeomorphism between the underlying spaces of any two geometric realizations is given by Corollary 4.11. □

4.14 Remark. From now on we will blur the distinction between abstract and geometric simplicial complexes using the approach which is most convenient at the time.

Simplicial homology groups

4.15 Definition. An *orientation* of an r-simplex $\langle v_0, v_1, \dots, v_r\rangle$ is an equivalence class of *orderings* of the vertices where two orderings are equivalent if and only if one can be obtained from the other by an even permutation. From now on we shall work with oriented simplices and write $\sigma_1 = \sigma_2$ if they are the same simplex with the same orientation and $\sigma_1 = -\sigma_2$ if they are the same simplex with opposite orientation.
Note. A 0-simplex $\langle v\rangle >$ has only one ordering of its vertex but we still consider it as having two orientations denoted by $\langle v\rangle$ and $-\langle v\rangle$.

4.16 Example. $\langle v_0, v_1, v_2, v_3\rangle = \langle v_1, v_0, v_3 v_2\rangle = \langle v_1, v_2, v_0, v_3\rangle = -\langle v_0, v_3, v_2, v_1\rangle$.

4.17 Definition. Suppose that K is a simplicial complex. For $r \in \mathbb{Z}$. the *r-chain group of K*, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of (non-empty) oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. An element of this group is called an *r-dimensional chain* of K.
Thus, for $r \geqslant 0$, $C_r(K) \cong \mathbb{Z}^k$ where k is the number of r-simplices in K. For $r < 0$, $C_r(K) = 0$.
An r-chain of K is an integral linear combination

$$\lambda_1\sigma_1 + \lambda_1\sigma_2 + \cdots + \lambda_q\sigma_q$$

where $\lambda_i \in \mathbb{Z}$ and $\sigma_i \in K_r$.

4.18 Definition. Suppose that K is a simplicial complex. For each $r \in \mathbb{Z}$ we define the *boundary homomorphism* $d_r \colon C_r(K) \to C_{r-1}(K)$ on the generators of $C_r(K)$ by

$$d_r(\langle v_0, v_1, \dots, v_r\rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_r\rangle$$

and then extend linearly.

Here $\hat{v}_i$ indicates that this vertex should be omitted.

Note that, by convention, $d_0(\langle v_0\rangle) = 0$.

4.19 Remark. It is not difficult (but a little tedious) to check that this definition is well-defined. For example, $\langle v_0, v_1, v_2\rangle = \langle v_1, v_2, v_0\rangle = -\langle v_0, v_2, v_1\rangle$. On applying the above formula these give:

$$d_2(\langle v_0, v_1, v_2\rangle) = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle,$$

$$d_2(\langle v_1, v_2, v_0\rangle) = \langle v_2, v_0\rangle - \langle v_1, v_0\rangle + \langle v_1, v_2\rangle,$$
$$d_2(-\langle v_0, v_2, v_1\rangle) = -\langle v_2, v_1\rangle + \langle v_0, v_1\rangle - \langle v_0, v_2\rangle,$$

and these values do all agree.

4.20 Definition. Given a simplicial complex K, the kernel of the boundary homomorphism $d_r\colon C_r(K) \to C_{r-1}(K)$ is called the *r-cycle group* of K and is denoted $Z_r(K)$. Thus

$$Z_r(K) = \{\, x \in C_r(K) \mid d_r(x) = 0 \,\}.$$

An element of $Z_r(K)$ is called an *r-dimensional cycle* (or just an r-cycle) of K.

4.21 Definition. Given a simplicial complex K, the image of the boundary homomorphism $d_{r+1}\colon C_{r+1}(K) \to C_r(K)$ is called the *r-boundary group* of K and is denoted $B_r(K)$. Thus

$$B_r(K) = \{\, x \in C_r(K) \mid x = d_{r+1}(y) \text{ for some } y \in C_{r+1}(K) \,\}.$$

An element of $B_r(K)$ is called an *r-dimensional boundary* (or just an r-boundary) of K.

4.22 Proposition. For a chain complex K, $d_r \circ d_{r+1} = 0\colon C_{r+1}(K) \to C_{r-1}(K)$.

The proof is given below.

4.23 Corollary. For a chain complex K, $B_r(K) \subset Z_r(K)$.

Proof. This is just a restatement of the Proposition. □

4.24 Definition. For a simplicial complex K, the *rth homology group*, $H_r(K)$ is defined to be the quotient group $Z_r(K)/B_r(K)$. An element of $H_r(K)$ is called an *r-dimensional homology class* of K.

4.25 Remarks. Recall that when H is a subgroup of the abelian group G, then we may define an equivalence relation on G by $g_1 \sim g_2 \Leftrightarrow g_1 - g_2 \in H$. The quotient group G/H has elements given by the equivalence classes $[g] = g + H = \{\, g + h \mid h \in H \,\}$ with composition given by $[g_1] + [g_2] = [g_1 + g_2]$ (see Definition G.6 in the background notes).
In the case of $G = Z_r(K)$, $H = B_r(K)$ this equivalence relation is called *homology*. We say that cycles z_1 and z_2 are *homologous*, written $z_1 \sim z_2$, when $z_1 - z_2 \in B_r(K)$. The cycle $z \in Z_r(K)$ represents the homology class $[z] = z + B_r(K) \in H_r(K)$.

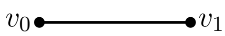
Thus a non-zero r-dimensional homology class of K is represented by an r-cycle which is not an r-boundary. Two r-cycles represent the same homology class if they differ by an r-boundary.

Proof of Proposition 4.22. Applying Definition 4.18,

$$\begin{aligned}
&d_r \circ d_{r+1}(\langle v_0, v_1, \ldots, v_{r+1}\rangle) \\
&= d_r\Big(\sum_{i=0}^{r+1}(-1)^i\langle v_0, \ldots, \hat{v}_i, \ldots, v_{r+1}\rangle\Big) \\
&= \sum_{i=0}^{r+1}(-1)^i d_r(\langle v_0, \ldots, \hat{v}_i, \ldots, v_{r+1}\rangle) \\
&= \sum_{i=0}^{r+1}(-1)^i\Big(\sum_{j=0}^{i-1}(-1)^j\langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{r+1}\rangle + \sum_{j=i}^{r}(-1)^j\langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_{j+1}, \ldots, v_{r+1}\rangle\Big) \\
&= \sum_{i=0}^{r+1}\sum_{j=0}^{i-1}(-1)^{i+j}\langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{r+1}\rangle + \sum_{i=0}^{r+1}\sum_{j=i}^{r}(-1)^{i+j}\langle v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_{j+1}, \ldots, v_{r+1}\rangle \\
&= 0
\end{aligned}$$

since the coefficient of $\langle v_0, \ldots, \hat{v}_k, \ldots, \hat{v}_l, \ldots, v_{r+1}\rangle$ is $(-1)^{k+l} + (-1)^{k+l-1} = (-1)^{k+l-1}(-1+1) = 0$. □

4.26 Example. Suppose that K is the following simplicial complex.



In other words $K = \{\langle v_0, v_1\rangle, \langle v_0\rangle, \langle v_1\rangle\}$.
Then $C_0(K) \cong \mathbb{Z}^2$ generated by $\langle v_0\rangle$ and $\langle v_1\rangle$ and $C_1(K) \cong \mathbb{Z}$ generated by $\langle v_0, v_1\rangle$. All other chain groups of K are trivial.
$d_1(\langle v_0, v_1\rangle) = \langle v_1\rangle - \langle v_0\rangle$ and all other boundary homomorphisms are zero.
Thus $Z_0(K) = C_0(K) \cong \mathbb{Z}^2$ generated by $\langle v_0\rangle$ and $\langle v_1\rangle$ and $B_0(K) \cong \mathbb{Z}$ generated by $\langle v_1\rangle - \langle v_0\rangle$ so that $\langle v_0\rangle \sim \langle v_1\rangle$.
Define $f\colon Z_0(K) \to \mathbb{Z}$ by $f(\lambda_0\langle v_0\rangle + \lambda_1\langle v_1\rangle) = \lambda_0 + \lambda_1$. Then f is a group epimorphism with $\mathrm{Ker}(f) = \{\lambda\langle v_0\rangle - \lambda\langle v_1\rangle\} = B_0(K)$.
Hence, by the First Isomorphism Theorem (Corollary G.9), f induces an isomorphism
$\overline{f}\colon H_0(K) = Z_0(K)/B_0(K) \to \mathbb{Z}$.
$Z_1(K) = \mathrm{Ker} d_1 = 0$ and so $H_1(K) = 0$.
There are no simplices of other dimensions and so all other homology groups are trivial. Thus

$$H_r(K) \cong \begin{cases} \mathbb{Z} & \text{for } r = 0, \\ 0 & \text{for } r \neq 0. \end{cases}$$

4.27 Proposition. Suppose that a simplicial complex K has dimension n. Then $H_r(K) = 0$ for $r < 0$ and $r > n$.

Proof. This is immediate from the definition since in these dimensions $C_r(K) = 0$ so that $Z_r(K) = B_r(K) = 0$ and so $H_r(K) = 0$. □

4.28 Definition. A simplicial complex is said to be *connected* when, for each pair of vertices in K there is a path along edges from one vertex to the other (cf. Definition 2.3(b)).

As in the proof of Proposition 2.5, If K is connected then the underlying space $|K|$ is path-connected.

4.29 Proposition. Suppose that the simplicial complex K is connected. Then $H_0(K) \cong \mathbb{Z}$.

Proof. Suppose that $V(K) = \{ v_1, v_2, \ldots, v_r \}$ so that $C_0(K) \cong \mathbb{Z}^r$.
Given the two vertices v_1 and v_s ($1 < s \leqslant r$), since K is connected, there is a sequence of edges, $\langle w_0, w_1 \rangle$, $\langle w_1, w_2 \rangle$, $\ldots$, $\langle w_{k-1}, w_k \rangle$ with $w_0 = v_1$ and $w_k = v_s$. Then

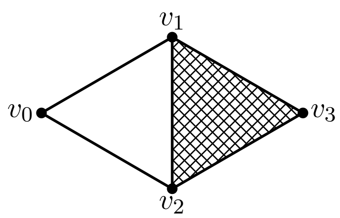
$$d_1(\langle w_0, w_1 \rangle + \langle w_1, w_2 \rangle + \cdots + \langle w_{k-1}, w_k \rangle) = \langle v_s \rangle - \langle v_1 \rangle.$$

So $\langle v_s \rangle - \langle v_1 \rangle \in B_0(K)$ for all $1 < s \leqslant r$, i.e. $\langle v_1 \rangle \sim \langle v_s \rangle$.
Now define $f \colon Z_0(K) = C_0(K) \to \mathbb{Z}$ by $f\left(\sum_{i=1}^r \lambda_i \langle v_i \rangle\right) = \sum_{i=1}^r \lambda_i$. Then

$$\operatorname{Ker}(f) = \left\{ \sum_{i=1}^r \lambda_i \langle v_i \rangle \;\middle|\; \sum_{i=1}^r \lambda_i = 0 \right\}$$

which is generated by the elements $\langle v_s \rangle - \langle v_1 \rangle$, $1 < s \leqslant r$, which also generate $B_0(K)$.
Hence $\operatorname{Ker}(f) = B_0(K)$ and so the First Isomorphism Theorem for groups gives an isomorphism $\overline{f} \colon H_0(K) = Z_0(K)/B_0(K) \to \mathbb{Z}$ and so $H_0(K) \cong \mathbb{Z}$ □.

4.30 Example. Suppose that K is the following simplicial complex.



In other words $K = \{ \langle v_1, v_2, v_3 \rangle, \langle v_0, v_1 \rangle, \langle v_0, v_2 \rangle, \langle v_1, v_2 \rangle, \langle v_1, v_3 \rangle, \langle v_2, v_3 \rangle, \langle v_0 \rangle, \langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle \}$.

Then $H_r(K) = 0$ for $r < 0$ and $r > 2$ since there are no simplices in these dimensions.

$H_0(K) \cong \mathbb{Z}$ since K is connected.

To find $Z_1(K)$ we solve

$$d_1(\lambda_1\langle v_0, v_1\rangle + \lambda_2\langle v_0, v_2\rangle + \lambda_3\langle v_1, v_2\rangle + \lambda_4\langle v_1, v_3\rangle + \lambda_5\langle v_2, v_3\rangle) = 0$$

and equating the coefficents of the vertices $\langle v_i\rangle$ to zero this gives the system

$$\begin{cases} -\lambda_1 - \lambda_2 \qquad = 0 \\ \lambda_1 \quad - \lambda_3 - \lambda_4 = 0 \\ \lambda_2 \quad + \lambda_3 - \lambda_5 = 0 \\ \lambda_4 \quad + \lambda_5 \qquad = 0 \end{cases} \iff \begin{cases} \lambda_1 + \lambda_2 \qquad = 0 \\ \lambda_2 + \lambda_3 + \lambda_4 = 0 \\ \lambda_4 + \lambda_5 \qquad = 0 \end{cases}$$

(reduced to row echelon form with free variables λ_3 and λ_5). So the solution space is spanned by $\{\,(1, -1, 1, 0, 0), (-1, 1, 0, -1, 1)\,\}$.

So $Z_1(K) \cong \mathbb{Z}^2$ is generated by

$$\begin{aligned} z_1 &= \langle v_0, v_1\rangle - \langle v_0, v_2\rangle + \langle v_1, v_2\rangle, \\ z_2 &= -\langle v_0, v_1\rangle + \langle v_0, v_2\rangle - \langle v_1, v_3\rangle + \langle v_2, v_3\rangle \end{aligned}$$

(which is pretty obvious from the above picture of K).

To find $B_1(K)$ we find

$$d_2(\langle v_1, v_2, v_3\rangle) = \langle v_2, v_3\rangle - \langle v_1, v_3\rangle + \langle v_1, v_2\rangle = z_1 + z_2.$$

So $B_1(K) \cong \mathbb{Z}$ generated by $z_1 + z_2$ so that $z_2 \sim -z_1$ or, more generally, $\lambda_1 z_1 + \lambda_2 z_2 \sim (\lambda_1 - \lambda_2)z_1$.

Hence $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}$ generated by $[z_1]$ (since the epimorphism $Z_1(K) \to \mathbb{Z}$ given by $\lambda_1 z_1 + \lambda_2 z_2 \mapsto \lambda_1 - \lambda_2$ has kernel $B_1(K)$).

Finally, since $d_2(\langle v_1, v_2, v_3\rangle) \neq 0$, $Z_2(K) = 0$ and so $H_2(K) = 0$.

Thus

$$H_r(K) \cong \begin{cases} \mathbb{Z} & \text{for } r = 0,\ 1, \\ 0 & \text{for } r \neq 0,\ 1. \end{cases}$$

4.31 Remarks. It is clear that a direct calculation as in the previous example is going to become very difficult once the number of simplices increases. There are several ways around this. One is to loosen the definition of simplicial complex so that we do not need to use so many simplices. This approach is taken by Hatcher who introduces the notion of Δ-set which is a generalization of the notion of simplicial complex needing many fewer simplices (for example you can triangulate the torus using only two 2-simplices, three edges and one vertex if Δ-sets are used). Another approach is to make use of a few general results from group theory to cut down the calculation and that is what is done in this course. One useful result is the Second Isomorphism Theorem (Proposition G.10) which is given in the background reading. Another is the following simple observation.

4.32 Proposition. Suppose that x, $x' \in C_r(K)$ are homologous r-chains in a simplicial complex K. Then, if one is a cycle, so is the other: $x \in Z_r(K) \Leftrightarrow x' \in Z_r(K)$.

Proof. If cycles $x, x' \in C_r(K)$ are homologous then $x - x' \in B_r(K) \subset Z_r(K)$ and so $d_r(x) - d_r(x') = d_r(x - x') = 0$ so that $d_r(x) = d_r(x')$. Thus $d_r(x) = 0 \Leftrightarrow d_r(x') = 0$, i.e. $x \in Z_r(K) \Leftrightarrow x' \in Z_r(K)$, $\square$

4.33 Example. Let K be the simplicial complex corresponding to the simplicial surface in Examples 2.8(b) so that $|K| \cong T_1$, the torus.
K has nine vertices $v_1, \ldots v_9$ and so $C_0(K) \cong \mathbb{Z}^9$.
K has 27 edges and so $C_1(K) \cong \mathbb{Z}^{27}$.
K has 18 2-simplices and so $C_2(K) \cong \mathbb{Z}^{18}$.
All other chain groups of K are trivial and so $H_i(K) \cong 0$ for $i < 0$ and for $i > 2$.
(a) $H_0(K) \cong \mathbb{Z}$ since K is a connected simplicial complex.
(b) *Calculation of* $H_1(K)$.
Step 1. Finding an expression for $Z_1(K)$.
Suppose that $x \in C_1(K)$. Then $x \sim x'$ where x' involves only edges corresponding the edges on the perimeter of the template and $\langle v_1, v_6\rangle$, $\langle v_1, v_8\rangle$ and $\langle v_7, v_9\rangle$, say.
To see this observe that all other edges can be eliminated by working through the 2-simplices in turn. For example, $d_2(\langle v_1, v_2, v_4\rangle) = \langle v_2, v_4\rangle - \langle v_1, v_4\rangle + \langle v_1, v_2\rangle \in B_1(K)$ and so $\langle v_2, v_4\rangle \sim \langle v_1, v_4\rangle - \langle v_1, v_2\rangle$. In this way eighteen edges can be eliminated leaving nine others.
However, from Proposition 4.32, $d_1(x) = 0 \Leftrightarrow d_1(x') = 0$ and so in this case the coefficients of $\langle v_1, v_6\rangle$, $\langle v_1, v_8\rangle$ and $\langle v_7, v_9\rangle$ must be zero.
Thus, if $x \in Z_1(K)$, then $x \sim x' \in Z_1(K)$ where

$$x' = \lambda_1\langle v_1, v_2\rangle + \lambda_2\langle v_1, v_3\rangle + \lambda_3\langle v_1, v_4\rangle + \lambda_4\langle v_1, v_7\rangle + \lambda_5\langle v_2, v_3\rangle + \lambda_6\langle v_4, v_7\rangle.$$

$$d_1(x') = -(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\langle v_1\rangle+(\lambda_1-\lambda_5)\langle v_2\rangle+(\lambda_2+\lambda_5)\langle v_3\rangle+(\lambda_3-\lambda_6)\langle v_4\rangle+(\lambda_4+\lambda_6)\langle v_7\rangle$$

and so, since $d_1(x') = 0$, $\lambda_1 = -\lambda_2 - \lambda_5$ and $\lambda_3 = -\lambda_4 = \lambda_6$.
Thus $x \in Z_1(K)$ implies that $x \sim x' \in V$, the subgroup generated by

$$x_1 = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$$

and

$$x_2 = \langle v_1, v_4\rangle + \langle v_4, v_7\rangle - \langle v_1, v_7\rangle$$

Hence $Z_1(K) = V + B_1(K)$.
Step 2. Find $V \cap B_1(K)$.
Suppose that $y \in C_2(K)$ and $d_2(y) \in V$. Then $d_2(y)$ involves no edges corresponding to the internal edges of the template which means that these edges must cancel out. Hence y is a multiple of the 2-chain

$$z = \langle v_1, v_2, v_4\rangle - \langle v_2, v_4, v_5\rangle + \langle v_2, v_3, v_5\rangle + \cdots$$

the sum of all of the 2-simplices oriented clockwise.
But $d_2(z) = x_1 + x_2 - x_1 - x_2 = 0$.
Hence $V \cap B_1(K) = 0$.
Step 3. Use the Second Isomorphism Theorem (Proposition G.10) to calculate $H_1(K)$ as follows.

$$H_1(K) = Z_1(K)/B_1(K) = \big(V+B_1(K)\big)/B_1(K) \cong V/\big(V\cap B_1(K)\big) \cong V/0 = V.$$

Thus $H_1(K) \cong V \cong \mathbb{Z}^2$ generated by $[x_1]$ and $[x_2]$.
(c) *Calculation of $H_2(K)$.*
If $y \in C_2(K)$ and $d_2(y) = 0$ then, as above y is a multiple of z. However, $d_2(z) = 0$. Hence $Z_2(K) \cong \mathbb{Z}$ generated by z.
$B_2(K) = 0$ since $C_3(K) = 0$.
Hence $H_2(K) = Z_2(K) \cong \mathbb{Z}$ generated by z.

Conclusion. $H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^2 & \text{for } i = 1, \\ \mathbb{Z} & \text{for } i = 2, \\ 0 & \text{otherwise.} \end{cases}$

4.34.1 Example. Let K be the simplicial complex corresponding to the simplicial surface in Example 2.8(c) so that $|K| \cong P_2$, the Klein bottle.
By a similar argument to Example 4.33, $Z_1(K) = B_1(K) + V$ where V is the free abelian group generated by

$$x_1 = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$$

and

$$x_2 = \langle v_1, v_4\rangle + \langle v_4, v_7\rangle - \langle v_1, v_7\rangle.$$

Now to find $V \cap B_1(K)$, if $y \in C_2(K)$ such that $d_2(y) \in V$ the the edges corresponding to the internal edges of the template must cancel out and so y must be a multiple of the 2-chain

$$z = \langle v_1, v_2, v_4\rangle + \langle v_2, v_5, v_4\rangle + \cdots$$

the sum of all of the 2-simplices oriented clockwise.
But $d_2(z) = x_1 - x_2 - x_1 - x_2 = -2x_2$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $2x_2$.
Thus

$$H_1(K) = Z_1(K)/B_1(K) = \big(V+B_1(K)\big)/B_1(K) \cong V/\big(V\cap B_1(K)\big) \cong \mathbb{Z}\times\mathbb{Z}_2$$

generated by $[x_1]$ and $[x_2]$, since the kernel of the homomorphism $f\colon V \to \mathbb{Z} \times \mathbb{Z}_2$ given by $\lambda_1 x_1 + \lambda_2 x_2 \mapsto (\lambda_1, [\lambda_2]_2)$ is generated by $2x_2$ and so is $V \cap B_1(K)$.

For $H_2(K)$ notice that, by the above argument, if $y \in Z_2(K)$ then, all of the edges corresponding to the internal edges of the template must cancel out and so y is a multiple of z. But $d_2(z) \neq 0$ and so $Z_2(K) = 0$ which means that $H_2(K) = 0$.

Conclusion. $H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z} \times \mathbb{Z}_2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$

4.34.2 Example. Let K be the simplicial complex corresponding to the simplicial surface in Example 2.8(d) so that $|K| \cong P^2$. Then

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Exercise)

Relationship with the Euler characteristic

4.35 Definition. The *rth Betti number.* $\beta_r(K)$ of a simplicial complex K is the rank of $H_r(K)$ (see Theorem G.14).

Remark. Note, that $\beta_r(K) = 0$ does *not* imply, that $H_r(K)$ is trivial, since the group might be a product of finite cyclic groups as it was the case for the projective plane and the Klein bottle.

4.36 Definition. The *Euler characteristic* of an n-dimensional simplicial complex K is given by $\chi(K) = \sum_{r=0}^{n} (-1)^r n_r$ where n_r is the number of r-simplices.

4.37 Theorem [the Euler-Poincaré Theorem]. For a simplicial complex K of dimension K,

$$\chi(K) = \sum_{r=0}^{n} (-1)^r \beta_r(K).$$

4.38 Remarks. (a) This shows that the homology groups determine the Euler characteristic. We can also show (see Problems 5) that a simplicial surface K is orientable if and only if $H_2(\bar{K}) = \mathbb{Z}$ and is non-orientable if and only if $H_2(\bar{K}) = 0$ and so the notion of orientability of a surface is also captured by the homology groups. The the homology groups are a real generalization of the two invariants of surfaces used in §3. If we can prove that the homology groups are topological invariants then it will follow that the Euler characteristic and whether a surface is orientable are topological invariants.

(b) The proof of Theorem 4.37 makes use of some notions relating to abelian groups which are of fundamental importance in algebraic topology and are summarized in the background notes. The key idea is that of *exact sequence* (Definition G.4).

Proof of Theorem 4.37. Recall that for a simplicial complex K,

$$B_{r-1}(K) = \mathrm{Im}(d_r \colon C_r(K) \to C_{r-1}(K)$$

and

$$Z_r(K) = \mathrm{Ker}(d_r \colon C_r(K) \to C_{r-1}(K).$$

Thus we have a short exact sequences

$$0 \to Z_r(K) \xrightarrow{i} C_r(K) \xrightarrow{d_r} B_{r-1}(K) \to 0. \tag{1}$$

Furthermore, $H_r(K) = Z_r(K)/B_r(K)$ by definition and so we have sequences

$$0 \to B_r(K) \to Z_r(K) \to H_r(K) \to 0. \tag{2}$$

Finally, $C_r(K)$ is the free abelian group on the r-simplices and so has rank n_r, the number of r-simplices.
We now apply Theorem G.14.
From short exact sequence (1) we deduce that

$$n_r = \mathrm{rank}\, Z_r(K) + \mathrm{rank}\, B_{r-1}(K). \tag{3}$$

And from short exact sequence (2) we deduce that

$$\mathrm{rank}\, Z_r(K) - \mathrm{rank}\, B_r(K) = \beta_r(K). \tag{4}$$

Hence,

$$\begin{aligned}
\chi(K) &= \sum_{r=0}^{n} (-1)^r n_r \quad \text{by definition} \\
&= \sum_{r=0}^{n} (-1)^r \big(\mathrm{rank}\, Z_r(K) + \mathrm{rank}\, B_{r-1}(K)\big) \quad \text{by (3)} \\
&= \sum_{r=0}^{n} (-1)^r \mathrm{rank}\, Z_r(K) + \sum_{r=0}^{n} (-1)^r \mathrm{rank}\, B_{r-1}(K) \\
&= \sum_{r=0}^{n} (-1)^r \mathrm{rank}\, Z_r(K) - \sum_{r=0}^{n} (-1)^r \mathrm{rank}\, B_r(K) \quad \text{since } B_r(K) = 0 \text{ for } r = -1, n \\
&= \sum_{r=0}^{n} (-1)^r \big(\mathrm{rank}\, Z_r(K) - \mathrm{rank}\, B_r(K)\big) \\
&= \sum_{r=0}^{n} (-1)^r \beta_r(K) \quad \text{by (4)}.
\end{aligned}$$

$\square$