Spring Semester 2020–2021

MATH31052 *Topology*

Problems 3: Subspaces and Product Spaces

1. Suppose that $X_2 \subset X_1 \subset X$ where X is a topological space. Prove that the subspace topology on X_2 induced by the topology on X is the same as the subspace topology on X_2 induced by the subspace topology on X_1.

2. Suppose that $X \subset \mathbb{R}^n$. Prove that the subspace topology on X induced by the usual topology on $\mathbb{R}^n$ is the same as the usual topology on X (given by Definition 2.5). [Remark 3.4]

3. Prove Proposition 3.5: For a subspace X_1 of a topological space X, the closed subsets of X_1 are all subsets of the form $A \cap X_1$ where A is a closed subset of X.

4. Prove that all open subsets of a subspace X_1 of a topological space X are open in X if and only if X_1 is open in X.
Deduce the following version of the Gluing Lemma (cf. Theorem 3.7). Suppose that X_1 and X_2 are open subspaces of X such that $X = X_1 \cup X_2$ and $f_i \colon X_i \to Y$ are continuous functions to a topological space Y ($i = 1, 2$) such that $f_1(x) = f_2(x)$ for $x \in X_1 \cap X_2$. Then the function $f \colon X \to Y$ defined by $f(x) = f_i(x)$ for $x \in X_i$ is well-defined and continuous.

5. Find subspaces X_1 and X_2 of a topological space X and continuous functions $f_i \colon X_i \to Y$, to a topological space Y such that $f_1(x) = f_2(x)$ for all $x \in X_1 \cap X_2$ such that the function $f \colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in X_i$ is not continuous. [Hint: What is the simplest non-continuous function you can think of? Find some subspaces on which its restriction is continuous.]

6. Prove that the product space $\mathbb{R} \times S^1$ (the infinite cylinder) is homeomorphic to the punctured plane $\mathbb{R}^2 - \{0\}$.

7. Given subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$ of topological spaces X_1 and X_2 prove that the following two topologies on $Y_1 \times Y_2$ are the same:

(i) $Y_1 \times Y_2$ is a subspace of the product space $X_1 \times X_2$;

(ii) $Y_1 \times Y_2$ is a product of the subspaces Y_1 (of X_1) and Y_2 (of X_2).

[Remark 3.12(c)]

[Hint: Consider the identity function $(Y_1 \times Y_2, \tau_1) \to (Y_1 \times Y_2, \tau_2)$ where τ_1 and τ_2 are the two topologies defined in the question. Use the universal properties of the product topology and the subspace topology to show that this identity function and its inverse are both continuous.]

8. Prove that the product space $\mathbb{R} \times \{0, 1\}$, where $\mathbb{R}$ has the usual topology and $\{0, 1\}$ has the indiscrete topology, is path-connected.

9. Suppose that X_1 and X_2 are disjoint topological spaces (i.e. $X_1 \cap X_2 = \emptyset$). Prove that we may define a topology on the union $X_1 \cup X_2$ by: $U \subset X_1 \cup X_2$ is open if and only if $U \cap X_1$ is open in X_1 and $U \cap X_2$ is open in X_2. With this topology $X_1 \cup X_2$ is called the *disjoint union* of the topological spaces (or sometimes the *coproduct*), often denoted $X_1 \sqcup X_2$.

Prove the following *universal property of the disjoint union topology.* A function $f\colon X_1 \sqcup X_2 \to Y$ to a topological space Y is continuous if and only if the restricted functions $f|X_1 = f \circ i_1\colon X_1 \to Y$ and $f|X_2 = f \circ i_2\colon X_2 \to Y$ are continuous (where $i_j\colon X_j \to X_1 \sqcup X_2$ for $j = 1, 2$ are the inclusion maps).